CHAPTER 2

Cost Concepts and Design Economics

The objective of Chapter 2 is to analyze short-term alternatives when the time value of money is not a factor. We accomplish this with three types of problems: 1) economic breakeven analysis; 2) cost-driven design optimization; and 3) present economy studies.

The A380 Superjumbo's Breakeven Point

When Europe's Airbus Company approved the A380 program in 2000, it was estimated that only 250 of the giant, 555-seat aircraft needed to be sold to break even. The program was initially based on expected deliveries of 751 aircraft over its life cycle. Long delays and mounting costs, however, have dramatically changed the original breakeven figure. In 2005, this figure was updated to 270 aircraft. According to an article in the *Financial Times* (October 20, 2006, p. 18), Airbus would have to sell 420 aircraft to break even—a 68% increase over the original estimate. To date, only 159 firm orders for the aircraft have been received. The topic of breakeven analysis is an integral part of this chapter.

The correct solution to any problem depends primarily on a true understanding of what the problem really is.

—Arthur M. Wellington (1887)

2.1 Cost Terminology

There are a variety of costs to be considered in an engineering economic analysis.* These costs differ in their frequency of occurrence, relative magnitude, and degree of impact on the study. In this section, we define a number of cost categories and illustrate how they should be treated in an engineering economic analysis.

2.1.1 Fixed, Variable, and Incremental Costs

Fixed costs are those unaffected by changes in activity level over a feasible range of operations for the capacity or capability available. Typical fixed costs include insurance and taxes on facilities, general management and administrative salaries, license fees, and interest costs on borrowed capital.

Of course, any cost is subject to change, but fixed costs tend to remain constant over a specific range of operating conditions. When larger changes in usage of resources occur, or when plant expansion or shutdown is involved, fixed costs can be affected.

Variable costs are those associated with an operation that varies in total with the quantity of output or other measures of activity level. For example, the costs of material and labor used in a product or service are variable costs, because they vary in total with the number of output units, even though the costs per unit stay the same.

An *incremental cost* (or *incremental revenue*) is the additional cost (or revenue) that results from increasing the output of a system by one (or more) units. Incremental cost is often associated with "go–no go" decisions that involve a limited change in output or activity level. For instance, the incremental cost per mile for driving an automobile may be $0.49, but this cost depends on considerations such as total mileage driven during the year (normal operating range), mileage expected for the next major trip, and the age of the automobile. Also, it is common to read about the "incremental cost of producing a barrel of oil" and "incremental cost to the state for educating a student." As these examples indicate, the incremental cost (or revenue) is often quite difficult to determine in practice.

EXAMPLE 2-1 Fixed and Variable Costs

In connection with surfacing a new highway, a contractor has a choice of two sites on which to set up the asphalt-mixing plant equipment. The contractor estimates that it will cost $1.15 per cubic yard mile (yd^3-mile) to haul the asphalt-paving material from the mixing plant to the job location. Factors relating to the two mixing sites are as follows (production costs at each site are the same):

* For the purposes of this book, the words *cost* and *expense* are used interchangeably.

Cost Factor	Site A	Site B
Average hauling distance	6 miles	4.3 miles
Monthly rental of site	$1,000	$5,000
Cost to set up and remove equipment	$15,000	$25,000
Hauling expense	$1.15/yd^3-mile	$1.15/yd^3-mile
Flagperson	Not required	$96/day

The job requires 50,000 cubic yards of mixed-asphalt-paving material. It is estimated that four months (17 weeks of five working days per week) will be required for the job. Compare the two sites in terms of their fixed, variable, and total costs. Assume that the cost of the return trip is negligible. Which is the better site? For the selected site, how many cubic yards of paving material does the contractor have to deliver before starting to make a profit if paid $8.05 per cubic yard delivered to the job location?

Solution

The fixed and variable costs for this job are indicated in the table shown next. Site rental, setup, and removal costs (and the cost of the flagperson at Site *B*) would be constant for the total job, but the hauling cost would vary in total amount with the distance and thus with the total output quantity of yd^3-miles (x).

Cost	Fixed	Variable	Site A	Site B
Rent	✓		= $4,000	= $20,000
Setup/removal	✓		= 15,000	= 25,000
Flagperson	✓		= 0	5(17)($96) = 8,160
Hauling		✓	6(50,000)($1.15) = 345,000	4.3(50,000)($1.15) = 247,250
			Total: $364,000	$300,410

Site *B*, which has the larger fixed costs, has the smaller total cost for the job. Note that the extra fixed costs of Site *B* are being "traded off" for reduced variable costs at this site.

The contractor will begin to make a profit at the point where total revenue equals total cost as a function of the cubic yards of asphalt pavement mix delivered. Based on Site *B*, we have

$$4.3(\$1.15) = \$4.945 \text{ in variable cost per yd}^3 \text{ delivered}$$

$$\text{Total cost} = \text{total revenue}$$

$$\$53{,}160 + \$4.945x = \$8.05x$$

$$x = 17{,}121 \text{ yd}^3 \text{ delivered.}$$

Therefore, by using Site *B*, the contractor will begin to make a profit on the job after delivering 17,121 cubic yards of material.

2.1.2 Direct, Indirect, and Standard Costs

These frequently encountered cost terms involve most of the cost elements that also fit into the previous overlapping categories of fixed and variable costs and recurring and nonrecurring costs. *Direct costs* are costs that can be reasonably measured and allocated to a specific output or work activity. The labor and material costs directly associated with a product, service, or construction activity are direct costs. For example, the materials needed to make a pair of scissors would be a direct cost.

Indirect costs are costs that are difficult to attribute or allocate to a specific output or work activity. Normally, they are costs allocated through a selected formula (such as proportional to direct labor hours, direct labor dollars, or direct material dollars) to the outputs or work activities. For example, the costs of common tools, general supplies, and equipment maintenance in a plant are treated as indirect costs.

Overhead consists of plant operating costs that are not direct labor or direct material costs. In this book, the terms *indirect costs*, *overhead*, and *burden* are used interchangeably. Examples of overhead include electricity, general repairs, property taxes, and supervision. Administrative and selling expenses are usually added to direct costs and overhead costs to arrive at a unit selling price for a product or service. (Appendix 2-A provides a more detailed discussion of cost accounting principles.)

Standard costs are planned costs per unit of output that are established in advance of actual production or service delivery. They are developed from anticipated direct labor hours, materials, and overhead categories (with their established costs per unit). Because total overhead costs are associated with a *certain level of production*, this is an important condition that should be remembered when dealing with standard cost data (for example, see Section 2.4.2). Standard costs play an important role in cost control and other management functions. Some typical uses are the following:

1. Estimating future manufacturing costs
2. Measuring operating performance by comparing actual cost per unit with the standard unit cost
3. Preparing bids on products or services requested by customers
4. Establishing the value of work in process and finished inventories

2.1.3 Cash Cost versus Book Cost

A cost that involves payment of cash is called a *cash cost* (and results in a cash flow) to distinguish it from one that does not involve a cash transaction and is reflected in the accounting system as a *noncash cost*. This noncash cost is often referred to as a *book cost*. Cash costs are estimated from the perspective established for the analysis (Principle 3, Section 1.2) and are the future expenses incurred for the alternatives being analyzed. Book costs are costs that do not involve cash payments but rather represent the recovery of past expenditures over a fixed period of time. The most common example of book cost is the *depreciation* charged for the use of assets such as plant and equipment. In engineering economic analysis, only those costs that are cash flows or potential cash flows from the defined perspective for the analysis need to be considered. *Depreciation, for example, is not a cash flow* and is important in an analysis only because it affects income taxes, which are cash flows. We discuss the topics of depreciation and income taxes in Chapter 7.

2.1.4 Sunk Cost

A *sunk cost* is one that has occurred in the past and has no relevance to estimates of future costs and revenues related to an alternative course of action. Thus, a sunk cost is common to all alternatives, is not part of the future (prospective) cash flows, and can be disregarded in an engineering economic analysis. For instance, sunk costs are nonrefundable cash outlays, such as earnest money on a house or money spent on a passport.

The concept of sunk cost is illustrated in the next simple example. Suppose that Joe College finds a motorcycle he likes and pays $40 as a down payment, which will be applied to the $1,300 purchase price, but which must be forfeited if he decides not to take the cycle. Over the weekend, Joe finds another motorcycle he considers equally desirable for a purchase price of $1,230. For the purpose of deciding which cycle to purchase, the $40 is a sunk cost and thus would not enter into the decision, except that it lowers the remaining cost of the first cycle. The decision then is between paying an additional $1,260 ($1,300 − $40) for the first motorcycle versus $1,230 for the second motorcycle.

In summary, sunk costs are irretrievable consequences of past decisions and therefore are irrelevant in the analysis and comparison of alternatives that affect the future. Even though it is sometimes emotionally difficult to do, sunk costs should be ignored, except possibly to the extent that their existence assists you to anticipate better what will happen in the future.

EXAMPLE 2-2 Sunk Costs in Replacement Analysis

A classic example of sunk cost involves the replacement of assets. Suppose that your firm is considering the replacement of a piece of equipment. It originally cost $50,000, is presently shown on the company records with a value of $20,000, and can be sold for an estimated $5,000. For purposes of replacement analysis, the $50,000 is a sunk cost. However, one view is that the sunk cost should be considered as the difference between the value shown in the company records and the present realizable selling price. According to this viewpoint, the sunk cost is $20,000 minus $5,000, or $15,000. Neither the $50,000 nor the $15,000, however, should be considered in an engineering economic analysis, except for the manner in which the $15,000 may affect income taxes, which will be discussed in Chapter 9.

2.1.5 Opportunity Cost

An *opportunity cost* is incurred because of the use of limited resources, such that the opportunity to use those resources to monetary advantage in an alternative use is foregone. Thus, it is the cost of the best rejected (i.e., foregone) opportunity and is often hidden or implied.

Consider a student who could earn $20,000 for working during a year, but chooses instead to go to school for a year and spend $5,000 to do so. The opportunity cost of going to school for that year is $25,000: $5,000 cash outlay and $20,000 for income foregone. (This figure neglects the influence of income taxes and assumes that the student has no earning capability while in school.)

EXAMPLE 2-3 Opportunity Cost in Replacement Analysis

The concept of an opportunity cost is often encountered in analyzing the replacement of a piece of equipment or other capital asset. Let us reconsider Example 2-2, in which your firm considered the replacement of an existing piece of equipment that originally cost $50,000, is presently shown on the company records with a value of $20,000, but has a present market value of only $5,000. For purposes of an engineering economic analysis of whether to replace the equipment, the present investment in that equipment should be considered as $5,000, because, by keeping the equipment, the firm is giving up the *opportunity* to obtain $5,000 from its disposal. Thus, the $5,000 immediate selling price is really the investment cost of not replacing the equipment and is based on the opportunity cost concept.

2.1.6 Life-Cycle Cost

In engineering practice, the term *life-cycle cost* is often encountered. This term refers to a summation of all the costs related to a product, structure, system, or service during its life span. The *life cycle* is illustrated in Figure 2-1. The life cycle begins with identification of the economic need or want (the requirement) and ends with retirement and disposal activities. It is a time horizon that must be defined in the context of the specific situation—whether it is a highway bridge, a jet engine for commercial aircraft, or an automated flexible manufacturing cell for a factory. The end of the life cycle may be projected on a functional or an economic basis. For example, the amount of time that a structure or piece of equipment is able to perform economically may be shorter than that permitted by its physical capability. Changes in the design efficiency of a boiler illustrate this situation. The old boiler may be able to produce the steam required, but not economically enough for the intended use.

The life cycle may be divided into two general time periods: the acquisition phase and the operation phase. As shown in Figure 2-1, each of these phases is further subdivided into interrelated but different activity periods.

The acquisition phase begins with an analysis of the economic need or want—the analysis necessary to make explicit the requirement for the product, structure, system, or service. Then, with the requirement explicitly defined, the other activities in the acquisition phase can proceed in a logical sequence. The conceptual design activities translate the defined technical and operational requirements into a preferred preliminary design. Included in these activities are development of the feasible alternatives and engineering economic analyses to assist in the selection of the preferred preliminary design. Also, advanced development and

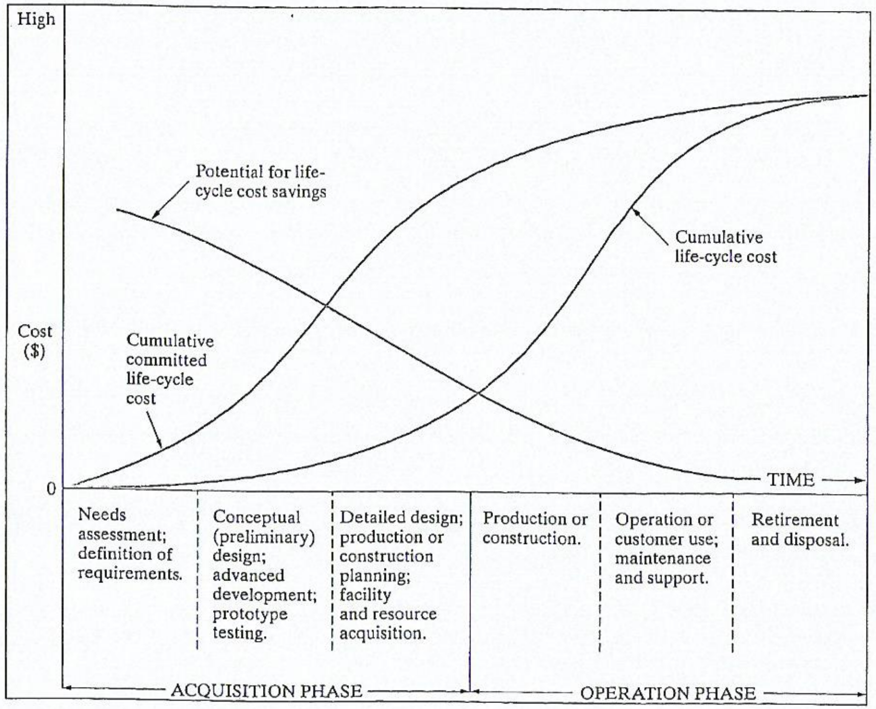


Figure 2-1 Phases of the Life Cycle and Their Relative Cost

prototype-testing activities to support the preliminary design work occur during this period.

The next group of activities in the acquisition phase involves detailed design and planning for production or construction. This step is followed by the activities necessary to prepare, acquire, and make ready for operation the facilities and other resources needed. *Again, engineering economy studies are an essential part of the design process to analyze and compare alternatives and to assist in determining the final detailed design.*

In the operation phase, the production, delivery, or construction of the end item(s) or service and their operation or customer use occur. This phase ends with retirement from active operation or use and, often, disposal of the physical assets involved. The priorities for engineering economy studies during the operation phase are (1) achieving efficient and effective support to operations, (2) determining whether (and when) replacement of assets should occur, and (3) projecting the timing of retirement and disposal activities.

Figure 2-1 shows relative cost profiles for the life cycle. The greatest potential for achieving life-cycle cost savings is early in the acquisition phase. How much of

the life-cycle costs for a product (for example) can be saved is dependent on many factors. However, effective engineering design and economic analysis during this phase are critical in maximizing potential savings.

The cumulative committed life-cycle cost curve increases rapidly during the acquisition phase. In general, approximately 80% of life-cycle costs are "locked in" at the end of this phase by the decisions made during requirements analysis and preliminary and detailed design. In contrast, as reflected by the cumulative life-cycle cost curve, only about 20% of actual costs occur during the acquisition phase, with about 80% being incurred during the operation phase.

Thus, one purpose of the life-cycle concept is to make explicit the interrelated effects of costs over the total life span for a product. An objective of the design process is to minimize the life-cycle cost—while meeting other performance requirements—by making the right trade-offs between prospective costs during the acquisition phase and those during the operation phase.

The cost elements of the life cycle that need to be considered will vary with the situation. Because of their common use, however, several basic life-cycle cost categories will now be defined.

The *investment cost* is the capital required for most of the activities in the acquisition phase. In simple cases, such as acquiring specific equipment, an investment cost may be incurred as a single expenditure. On a large, complex construction project, however, a series of expenditures over an extended period could be incurred. This cost is also called a *capital investment*.

The term *working capital* refers to the funds required for current assets (i.e., other than fixed assets such as equipment, facilities, etc.) that are needed for the start-up and support of operational activities. For example, products cannot be made or services delivered without having materials available in inventory. Functions such as maintenance cannot be supported without spare parts, tools, trained personnel, and other resources. Also, cash must be available to pay employee salaries and the other expenses of operation. The amount of working capital needed will vary with the project involved, and some or all of the investment in working capital is usually recovered at the end of a project's life.

Operation and maintenance cost (O&M) includes many of the recurring annual expense items associated with the operation phase of the life cycle. The direct and indirect costs of operation associated with the five primary resource areas—people, machines, materials, energy, and information—are a major part of the costs in this category.

Disposal cost includes those nonrecurring costs of shutting down the operation and the retirement and disposal of assets at the end of the life cycle. Normally, costs associated with personnel, materials, transportation, and one-time special activities can be expected. These costs will be offset in some instances by receipts from the sale of assets with remaining market value. A classic example of a disposal cost is that associated with cleaning up a site where a chemical processing plant had been located.

2.2 The General Economic Environment

There are numerous general economic concepts that must be taken into account in engineering studies. In broad terms, economics deals with the interactions between people and wealth, and engineering is concerned with the cost-effective use of scientific knowledge to benefit humankind. This section introduces some of these basic economic concepts and indicates how they may be factors for consideration in engineering studies and managerial decisions.

2.2.1 Consumer and Producer Goods and Services

The goods and services that are produced and utilized may be divided conveniently into two classes. *Consumer goods and services* are those products or services that are directly used by people to satisfy their wants. Food, clothing, homes, cars, television sets, haircuts, opera, and medical services are examples. The providers of consumer goods and services must be aware of, and are subject to, the changing wants of the people to whom their products are sold.

Producer goods and services are used to produce consumer goods and services or other producer goods. Machine tools, factory buildings, buses, and farm machinery are examples. The amount of producer goods needed is determined indirectly by the amount of consumer goods or services that are demanded by people. However, because the relationship is much less direct than for consumer goods and services, the demand for and production of producer goods may greatly precede or lag behind the demand for the consumer goods that they will produce.

2.2.2 Measures of Economic Worth

Goods and services are produced and desired because they have *utility*—the power to satisfy human wants and needs. Thus, they may be used or consumed directly, or they may be used to produce other goods or services. Utility is most commonly measured in terms of *value*, expressed in some medium of exchange as the *price* that must be paid to obtain the particular item.

Much of our business activity, including engineering, focuses on increasing the utility (value) of materials and products by changing their form or location. Thus, iron ore, worth only a few dollars per ton, significantly increases in value by being processed, combined with suitable alloying elements, and converted into razor blades. Similarly, snow, worth almost nothing when high in distant mountains, becomes quite valuable when it is delivered in melted form several hundred miles away to dry southern California.

2.2.3 Necessities, Luxuries, and Price Demand

Goods and services may be divided into two types: *necessities* and *luxuries*. Obviously, these terms are relative, because, for most goods and services, what one person considers a necessity may be considered a luxury by another. For example, a person living in one community may find that an automobile is a necessity to get

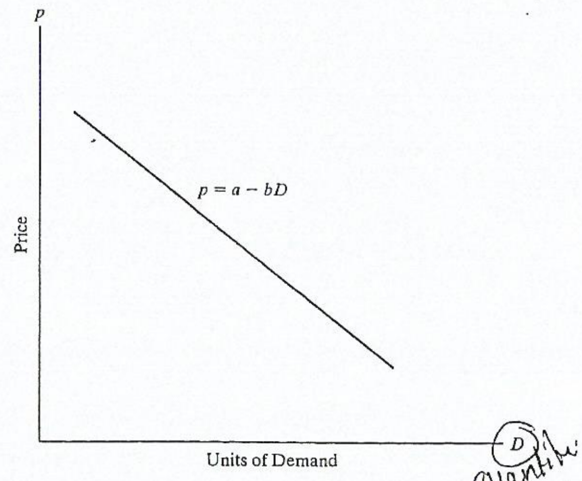


Figure 2-2 General Price–Demand Relationship. (Note that price is considered to be the independent variable but is shown as the vertical axis. This convention is commonly used by economists.)

to and from work. If the same person lived and worked in a different city, adequate public transportation might be available, and an automobile would be a luxury. For all goods and services, there is a relationship between the price that must be paid and the quantity that will be demanded or purchased. This general relationship is depicted in Figure 2-2. As the selling price per unit (p) is increased, there will be less demand (D) for the product, and as the selling price is decreased, the demand will increase. The relationship between price and demand can be expressed as the linear function

$$p = a - bD \qquad \text{for } 0 \le D \le \frac{a}{b}, \text{ and } a > 0,\ b > 0, \tag{2-1}$$

where a is the intercept on the price axis and $-b$ is the slope. Thus, b is the amount by which demand increases for each unit decrease in p. Both a and b are constants. It follows, of course, that

$$D = \frac{a - p}{b} \qquad (b \neq 0). \tag{2-2}$$

2.2.4 Competition

Because economic laws are general statements regarding the interaction of people and wealth, they are affected by the economic environment in which people and wealth exist. Most general economic principles are stated for situations in which *perfect competition* exists.

Perfect competition occurs in a situation in which any given product is supplied by a large number of vendors and there is no restriction on additional suppliers entering the market. Under such conditions, there is assurance of complete freedom on the part of both buyer and seller. Perfect competition may never occur in actual practice, because of a multitude of factors that impose some degree of limitation upon the actions of buyers or sellers, or both. However, with conditions of perfect competition assumed, it is easier to formulate general economic laws.

Monopoly is at the opposite pole from perfect competition. A perfect monopoly exists when a unique product or service is only available from a single supplier and that vendor can prevent the entry of all others into the market. Under such conditions, the buyer is at the complete mercy of the supplier in terms of the availability and price of the product. Perfect monopolies rarely occur in practice, because (1) few products are so unique that substitutes cannot be used satisfactorily and (2) governmental regulations prohibit monopolies if they are unduly restrictive.

2.2.5 The Total Revenue Function

The total revenue, TR, that will result from a business venture during a given period is the product of the selling price per unit, p, and the number of units sold, D. Thus,

$$\text{TR} = \text{price} \times \text{demand} = p \cdot D. \tag{2-3}$$

If the relationship between price and demand as given in Equation (2-1) is used,

$$\text{TR} = (a - bD)D = aD - bD^2 \quad \text{for } 0 \le D \le \frac{a}{b} \text{ and } a > 0, b > 0. \tag{2-4}$$

The relationship between total revenue and demand for the condition expressed in Equation (2-4) may be represented by the curve shown in Figure 2-3. From calculus, the demand, $\hat{D}$, that will produce maximum total revenue can be obtained by solving

$$\frac{d\text{TR}}{dD} = a - 2bD = 0. \tag{2-5}$$

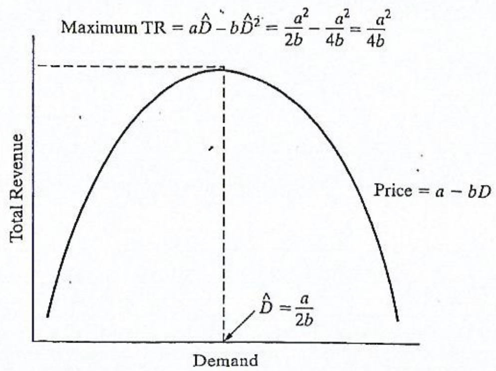


Figure 2-3 Total Revenue Function as a Function of Demand

Thus,*

$$\hat{D} = \frac{a}{2b}. \tag{2-6}$$

It must be emphasized that, because of cost–volume relationships (discussed in the next section), *most businesses would not obtain maximum profits by maximizing revenue.* Accordingly, the cost–volume relationship must be considered and related to revenue, because cost reductions provide a key motivation for many engineering process improvements.

2.2.6 Cost, Volume, and Breakeven Point Relationships

Fixed costs remain constant over a wide range of activities, but variable costs vary in total with the volume of output (Section 2.1.1). Thus, at any demand D, total cost is

$$C_T = C_F + C_V, \tag{2-7}$$

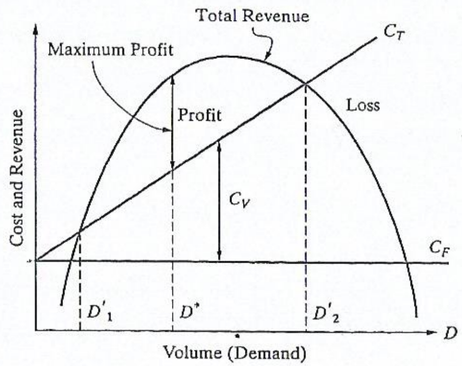
where C_F and C_V denote fixed and variable costs, respectively. For the linear relationship assumed here,

$$C_V = c_v \cdot D, \tag{2-8}$$

where c_v is the variable cost per unit. In this section, we consider two scenarios for finding breakeven points. In the first scenario, demand is a function of price. The second scenario assumes that price and demand are independent of each other.

Scenario 1 When total revenue, as depicted in Figure 2-3, and total cost, as given by Equations (2-7) and (2-8), are combined, the typical results as a function of demand are depicted in Figure 2-4. At *breakeven point* D'_1, total revenue is equal

Figure 2-4 Combined Cost and Revenue Functions, and Breakeven Points, as Functions of Volume, and Their Effect on Typical Profit (Scenario 1)



* To guarantee that $\hat{D}$ maximizes total revenue, check the second derivative to be sure it is negative:

$$\frac{d^2\text{TR}}{dD^2} = -2b.$$

Also, recall that in cost-minimization problems, a positively signed second derivative is necessary to guarantee a minimum-value optimal cost solution.

to total cost, and an increase in demand will result in a profit for the operation. Then at optimal demand, D^*, profit is maximized [Equation (2-10)]. At breakeven point D'_2, total revenue and total cost are again equal, but additional volume will result in an operating loss instead of a profit. Obviously, the conditions for which breakeven and maximum profit occur are our primary interest. First, at any volume (demand), D,

$$\text{Profit (loss)} = \text{total revenue} - \text{total costs}$$
$$= (aD - bD^2) - (C_F + c_v D)$$
$$= -bD^2 + (a - c_v)D - C_F \quad \text{for } 0 \le D \le \frac{a}{b} \text{ and } a > 0,\ b > 0. \qquad (2\text{-}9)$$

In order for a profit to occur, based on Equation (2-9), and to achieve the typical results depicted in Figure 2-4, two conditions must be met:

1. $(a - c_v) > 0$; that is, the price per unit that will result in no demand has to be greater than the variable cost per unit. (This avoids negative demand.)
2. TR must exceed total cost (C_T) for the period involved.

If these conditions are met, we can find the optimal demand at which maximum profit will occur by taking the first derivative of Equation (2-9) with respect to D and setting it equal to zero:

$$\frac{d(\text{profit})}{dD} = a - c_v - 2bD = 0.$$

The optimal value of D that maximizes profit is

$$D^* = \frac{a - c_v}{2b}. \qquad (2\text{-}10)$$

To ensure that we have *maximized* profit (rather than minimized it), the sign of the second derivative must be negative. Checking this, we find that

$$\frac{d^2(\text{profit})}{dD^2} = -2b,$$

which will be negative for $b > 0$ (as specified earlier).

An economic breakeven point for an operation occurs when total revenue equals total cost. Then for total revenue and total cost, as used in the development of Equations (2-9) and (2-10) and at any demand D,

$$\text{Total revenue} = \text{total cost} \quad \text{(breakeven point)}$$
$$aD - bD^2 = C_F + c_v D$$
$$-bD^2 + (a - c_v)D - C_F = 0. \qquad (2\text{-}11)$$

Because Equation (2-11) is a quadratic equation with one unknown (D), we can solve for the breakeven points D'_1 and D'_2 (the roots of the equation):*

$$D' = \frac{-(a - c_v) \pm [(a - c_v)^2 - 4(-b)(-C_F)]^{1/2}}{2(-b)}. \qquad (2\text{-}12)$$

With the conditions for a profit satisfied [Equation (2-9)], the quantity in the brackets of the numerator (the discriminant) in Equation (2-12) will be greater than zero. This will ensure that D'_1 and D'_2 have real positive, unequal values.

EXAMPLE 2-4

Optimal Demand When Demand Is a Function of Price

A company produces an electronic timing switch that is used in consumer and commercial products. The fixed cost (C_F) is \$73,000 per month, and the variable cost (c_v) is \$83 per unit. The selling price per unit is $p = \$180 - 0.02(D)$, based on Equation (2-1). For this situation,

(a) determine the optimal volume for this product and confirm that a profit occurs (instead of a loss) at this demand.

(b) find the volumes at which breakeven occurs; that is, what is the range of profitable demand? Solve by hand and by spreadsheet.

Solution by Hand

(a) $D^* = \dfrac{a - c_v}{2b} = \dfrac{\$180 - \$83}{2(0.02)} = 2{,}425$ units per month [from Equation (2-10)].

Is $(a - c_v) > 0$?

$$(\$180 - \$83) = \$97, \quad \text{which is greater than 0.}$$

And is (total revenue − total cost) > 0 for $D^* = 2{,}425$ units per month?

$$[\$180(2{,}425) - 0.02(2{,}425)^2] - [\$73{,}000 + \$83(2{,}425)] = \$44{,}612$$

A demand of $D^* = 2{,}425$ units per month results in a maximum profit of \$44,612 per month. Notice that the second derivative is negative (−0.04).

(b) Total revenue = total cost (breakeven point)

$$-bD^2 + (a - c_v)D - C_F = 0 \qquad \text{[from Equation (2-11)]}$$
$$-0.02D^2 + (\$180 - \$83)D - \$73{,}000 = 0$$
$$-0.02D^2 + 97D - 73{,}000 = 0$$

* Given the quadratic equation $ax^2 + bx + c = 0$, the roots are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

And, from Equation (2-12),

$$D' = \frac{-97 \pm [(97)^2 - 4(-0.02)(-73{,}000)]^{0.5}}{2(-0.02)}$$

$$D_1' = \frac{-97 + 59.74}{-0.04} = 932 \text{ units per month}$$

$$D_2' = \frac{-97 - 59.74}{-0.04} = 3{,}918 \text{ units per month.}$$

Thus, the range of profitable demand is 932–3,918 units per month.

Spreadsheet Solution

Figure 2-5(a) displays the spreadsheet solution for this problem. This spreadsheet calculates profit for a range of demand values (shown in column A). For a specific value of demand, price per unit is calculated in column B by using Equation (2-1) and Total Revenue is simply demand × price. Total Expense is computed by using Equations (2-7) and (2-8). Finally, Profit (column E) is then Total Revenue − Total Expense.

A quick inspection of the Profit column gives us an idea of the optimal demand value as well as the breakeven points. Note that profit steadily increases as demand increases to 2,500 units per month and then begins to drop off. This tells us that the optimal demand value lies in the range of 2,250 to 2,750 units per month. A more specific value can be obtained by changing the Demand Start point value in cell E1 and the Demand Increment value in cell E2. For example, if the value of cell E1 is set to 2,250 and the increment in cell E2 is set to 10, the optimal demand value is shown to be between 2,420 and 2,430 units per month.

The breakeven points lie within the ranges 750–1,000 units per month and 3,750–4,000 units per month, as indicated by the change in sign of profit. Again, by changing the values in cells E1 and E2, we can obtain more exact values of the breakeven points.

Figure 2-5(b) is a graphical display of the Total Revenue, Total Expense, and Profit functions for the range of demand values given in column A of Figure 2-5(a). This graph enables us to see how profit changes as demand increases. The optimal demand value (maximum point of the profit curve) appears to be around 2,500 units per month.

Figure 2-5(b) is also a graphical representation of the breakeven points. By graphing the total revenue and total cost curves separately, we can easily identify the breakeven points (the intersection of these two functions). From the graph, the range of profitable demand is approximately 1,000 to 4,000 units per month. Notice also that, at these demand values, the profit curve crosses the x-axis ($0).

= B3 − B4 * A7

= B1 + B2 * A7

= B7 * A7

= C7 − D7

= E1

	A	B	C	D	E
1	Fixed cost/ mo. =	$ 73,000		Demand Start point (D) =	0
2	Variable cost/unit =	$ 83		Demand Increment =	250
3	a =	$ 180			
4	b =	$ 0.02			
5					
6	**Monthly Demand**	**Price per Unit**	**Total Revenue**	**Total Expense**	**Profit**
7	0	$ 180	$ -	$ 73,000	$ (73,000)
8	250	$ 175	$ 43,750	$ 93,750	$ (50,000)
9	500	$ 170	$ 85,000	$ 114,500	$ (29,500)
10	750	$ 165	$ 123,750	$ 135,250	$ (11,500)
11	1000	$ 160	$ 160,000	$ 156,000	$ 4,000
12	1250	$ 155	$ 193,750	$ 176,750	$ 17,000
13	1500	$ 150	$ 225,000	$ 197,500	$ 27,500
14	1750	$ 145	$ 253,750	$ 218,250	$ 35,500
15	2000	$ 140	$ 280,000	$ 239,000	$ 41,000
16	2250	$ 135	$ 303,750	$ 259,750	$ 44,000
17	2500	$ 130	$ 325,000	$ 280,500	$ 44,500
18	2750	$ 125	$ 343,750	$ 301,250	$ 42,500
19	3000	$ 120	$ 360,000	$ 322,000	$ 38,000
20	3250	$ 115	$ 373,750	$ 342,750	$ 31,000
21	3500	$ 110	$ 385,000	$ 363,500	$ 21,500
22	3750	$ 105	$ 393,750	$ 384,250	$ 9,500
23	4000	$ 100	$ 400,000	$ 405,000	$ (5,000)
24	4250	$ 95	$ 403,750	$ 425,750	$ (22,000)
25	4500	$ 90	$ 405,000	$ 446,500	$ (41,500)
26	4750	$ 85	$ 403,750	$ 467,250	$ (63,500)
27	5000	$ 80	$ 400,000	$ 488,000	$ (88,000)
28	5250	$ 75	$ 393,750	$ 508,750	$ (115,000)
29	5500	$ 70	$ 385,000	$ 529,500	$ (144,500)

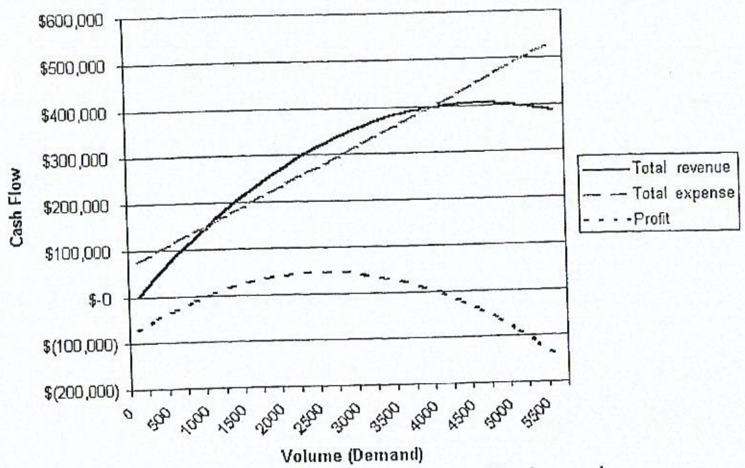
= A7 + E2

(a) Table of profit values for a range of demand values

Figure 2-5 Spreadsheet Solution, Example 2-4

Comment

As seen in the hand solution to this problem, Equations (2-10) and (2-12) can be used directly to solve for the optimal demand value and breakeven points.



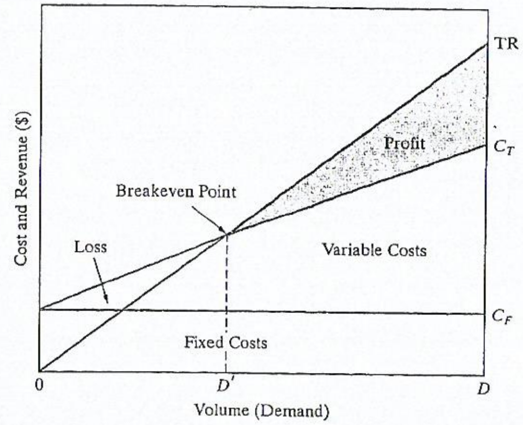
(b) Graphical display of optimal demand and breakeven values

Figure 2-5 *(continued)*

The power of the spreadsheet in this example is the ease with which graphical displays can be generated to support your analysis. Remember, a picture really can be worth a thousand words. Spreadsheets also facilitate sensitivity analysis (to be discussed more fully in Chapter 11). For example, what is the impact on the optimal demand value and breakeven points if variable costs are reduced by 10% per unit? (The new optimal demand value is increased to 2,632 units per month, and the range of profitable demand is widened to 822 to 4,443 units per month.)

Scenario 2 When the price per unit (p) for a product or service can be represented more simply as being independent of demand [versus being a linear function of demand, as assumed in Equation (2-1)] and is greater than the variable cost per unit (c_v), a single breakeven point results. Then, under the assumption that demand is immediately met, total revenue (TR) $= p \cdot D$. If the linear relationship for costs in Equations (2-7) and (2-8) is also used in the model, the typical situation is depicted in Figure 2-6. This scenario is typified by the Airbus example presented at the beginning of the chapter.

Figure 2-6 Typical Breakeven Chart with Price (p) a Constant (Scenario 2)



EXAMPLE 2-5 Breakeven Point When Price Is Independent of Demand

An engineering consulting firm measures its output in a standard service hour unit, which is a function of the personnel grade levels in the professional staff. The variable cost (c_v) is $62 per standard service hour. The charge-out rate [i.e., selling price (p)] is $85.56 per hour. The maximum output of the firm is 160,000 hours per year, and its fixed cost (C_F) is $2,024,000 per year. For this firm,

(a) what is the breakeven point in standard service hours and in percentage of total capacity?

(b) what is the percentage reduction in the breakeven point (sensitivity) if fixed costs are reduced 10%; if variable cost per hour is reduced 10%; and if the selling price per unit is increased by 10%?

Solution

(a)

$$\text{Total revenue} = \text{total cost} \qquad \text{(breakeven point)}$$

$$pD' = C_F + c_v D'$$

$$D' = \frac{C_F}{(p - c_v)}, \qquad (2\text{-}13)$$

and

$$D' = \frac{\$2,024,000}{(\$85.56 - \$62)} = 85,908 \text{ hours per year}$$

$$D' = \frac{85,908}{160,000} = 0.537,$$

or 53.7% of capacity.

(b) A 10% reduction in C_F gives

$$D' = \frac{0.9(\$2{,}024{,}000)}{(\$85.56 - \$62)} = 77{,}318 \text{ hours per year}$$

and

$$\frac{85{,}908 - 77{,}318}{85{,}908} = 0.10,$$

or a 10% reduction in D'.

A 10% reduction in c_v gives

$$D' = \frac{\$2{,}024{,}000}{[\$85.56 - 0.9(\$62)]} = 68{,}011 \text{ hours per year}$$

and

$$\frac{85{,}908 - 68{,}011}{85{,}908} = 0.208,$$

or a 20.8% reduction in D'.

A 10% increase in p gives

$$D' = \frac{\$2{,}024{,}000}{[1.1(\$85.56) - \$62]} = 63{,}021 \text{ hours per year}$$

and

$$\frac{85{,}908 - 63{,}021}{85{,}908} = 0.266,$$

or a 26.6% reduction in D'.

Thus, the breakeven point is more sensitive to a reduction in variable cost per hour than to the same percentage reduction in the fixed cost. Furthermore, notice that the breakeven point in this example is highly sensitive to the selling price per unit, p.

Market competition often creates pressure to lower the breakeven point of an operation; the lower the breakeven point, the less likely that a loss will occur during market fluctuations. Also, if the selling price remains constant (or increases), a larger profit will be achieved at any level of operation above the reduced breakeven point.

2.3 Cost-Driven Design Optimization

As discussed in Section 2.1.6, engineers must maintain a *life-cycle* (i.e., "cradle to grave") viewpoint as they design products, processes, and services. Such a complete perspective ensures that engineers consider initial investment costs,

operation and maintenance expenses and other annual expenses in later years, and environmental and social consequences over the life of their designs. In fact, a movement called *Design for the Environment* (DFE), or "green engineering," has prevention of waste, improved materials selection, and reuse and recycling of resources among its goals. Designing for energy conservation, for example, is a subset of green engineering. Another example is the design of an automobile bumper that can be easily recycled. As you can see, *engineering design is an economically driven art.*

Examples of cost minimization through effective design are plentiful in the practice of engineering. Consider the design of a heat exchanger in which tube material and configuration affect cost and dissipation of heat. The problems in this section designated as "cost-driven design optimization" are simple design models intended to illustrate the importance of cost in the design process. These problems show the procedure for determining an optimal design, using cost concepts. We will consider discrete and continuous optimization problems that involve a single design variable, X. This variable is also called a *primary cost driver*, and knowledge of its behavior may allow a designer to account for a large portion of total cost behavior.

For cost-driven design optimization problems, the two main tasks are as follows:

1. Determine the optimal value for a certain alternative's design variable. For example, what velocity of an aircraft minimizes the total annual costs of owning and operating the aircraft?
2. Select the best alternative, each with its own unique value for the design variable. For example, what insulation thickness is best for a home in Virginia: R11, R19, R30, or R38?

In general, the cost models developed in these problems consist of three types of costs:

1. fixed cost(s)
2. cost(s) that vary *directly* with the design variable
3. cost(s) that vary *indirectly* with the design variable

A simplified format of a cost model with one design variable is

$$\text{Cost} = aX + \frac{b}{X} + k, \tag{2-14}$$

where a is a parameter that represents the directly varying cost(s),
b is a parameter that represents the indirectly varying cost(s),
k is a parameter that represents the fixed cost(s), and
X represents the design variable in question (e.g., weight or velocity).

In a particular problem, the parameters a, b, and k may actually represent the sum of a group of costs in that category, and the design variable may be raised to some power for either directly or indirectly varying costs.*

The following steps outline a general approach for optimizing a design with respect to cost:

1. Identify the design variable that is the primary cost driver (e.g., pipe diameter or insulation thickness).
2. Write an expression for the cost model in terms of the design variable.
3. Set the first derivative of the cost model with respect to the continuous design variable equal to zero. For discrete design variables, compute the value of the cost model for each discrete value over a selected range of potential values.
4. Solve the equation found in Step 3 for the optimum value of the continuous design variable.† For discrete design variables, the optimum value has the minimum cost value found in Step 3. This method is analogous to taking the first derivative for a continuous design variable and setting it equal to zero to determine an optimal value.
5. For continuous design variables, use the second derivative of the cost model with respect to the design variable to determine whether the optimum value found in Step 4 corresponds to a global maximum or minimum.

EXAMPLE 2-6 How Fast Should the Airplane Fly?

The cost of operating a jet-powered commercial (passenger-carrying) airplane varies as the three-halves (3/2) power of its velocity; specifically, $C_O = knv^{3/2}$, where n is the trip length in miles, k is a constant of proportionality, and v is velocity in miles per hour. It is known that at 400 miles per hour, the *average* cost of operation is \$300 per mile. The company that owns the aircraft wants to minimize the cost of operation, but that cost must be balanced against the cost of the passengers' time (C_C), which has been set at \$300,000 per hour.

(a) At what velocity should the trip be planned to minimize the total cost, which is the sum of the cost of operating the airplane and the cost of passengers' time?

(b) How do you know that your answer for the problem in Part (a) minimizes the total cost?

* A more general model is the following: Cost $= k + ax + b_1x^{e_1} + b_2x^{e_2} + \cdots$, where $e_1 = -1$ reflects costs that vary inversely with X, $e_2 = 2$ indicates costs that vary as the square of X, and so forth.

† If multiple optima (stationary points) are found in Step 4, finding the global optimum value of the design variable will require a little more effort. One approach is to systematically use each root in the second derivative equation and assign each point as a maximum or a minimum based on the sign of the second derivative. A second approach would be to use each root in the objective function and see which point best satisfies the cost function.

Solution

(a) The equation for total cost (C_T) is

$$C_T = C_O + C_C = knv^{3/2} + (\$300{,}000 \text{ per hour})\left(\frac{n}{v}\right),$$

where n/v has time (hours) as its unit.

Now we solve for the value of k:

$$\frac{C_O}{n} = kv^{3/2}$$

$$\frac{\$300}{\text{mile}} = k\left(400\frac{\text{miles}}{\text{hour}}\right)^{3/2}$$

$$k = \frac{\$300/\text{mile}}{\left(400\dfrac{\text{miles}}{\text{hour}}\right)^{3/2}}$$

$$k = \frac{\$300/\text{mile}}{8000\left(\dfrac{\text{miles}^{3/2}}{\text{hour}^{3/2}}\right)}$$

$$k = \$0.0375\frac{\text{hours}^{3/2}}{\text{miles}^{5/2}}.$$

Thus,

$$C_T = \left(\$0.0375\frac{\text{hours}^{3/2}}{\text{miles}^{5/2}}\right)(n \text{ miles})\left(v\frac{\text{miles}}{\text{hour}}\right)^{3/2} + \left(\frac{\$300{,}000}{\text{hour}}\right)\left(\frac{n \text{ miles}}{v\dfrac{\text{miles}}{\text{hour}}}\right)$$

$$C_T = \$0.0375nv^{3/2} + \$300{,}000\left(\frac{n}{v}\right).$$

Next, the first derivative is taken:

$$\frac{dC_T}{dv} = \frac{3}{2}(\$0.0375)nv^{1/2} - \frac{\$300{,}000n}{v^2} = 0.$$

So,

$$0.05625v^{1/2} - \frac{300{,}000}{v^2} = 0$$

$$0.05625v^{5/2} - 300{,}000 = 0$$

$$v^{5/2} = \frac{300{,}000}{0.05625} = 5{,}333{,}333$$

$$v^* = (5{,}333{,}333)^{0.4} = 490.68 \text{ mph}.$$

(b) Finally, we check the second derivative to confirm a minimum cost solution:

$$\frac{d^2C_T}{dv^2} = \frac{0.028125}{v^{1/2}} + \frac{600{,}000}{v^3} \quad \text{for } v > 0, \text{ and therefore, } \frac{d^2C_T}{dv^2} > 0.$$

The company concludes that $v = 490.68$ mph minimizes the total cost of this particular airplane's flight.

EXAMPLE 2.7

Energy Savings through Increased Insulation

This example deals with a discrete optimization problem of determining the most economical amount of attic insulation for a large single-story home in Virginia. In general, the heat lost through the roof of a single-story home is

$$\begin{array}{c}\text{Heat loss}\\ \text{in Btu}\\ \text{per hour}\end{array} = \left(\begin{array}{c}\Delta\ \text{Temperature}\\ \text{in °F}\end{array}\right)\left(\begin{array}{c}\text{Area}\\ \text{in}\\ \text{ft}^2\end{array}\right)\left(\begin{array}{c}\text{Conductance in}\\ \dfrac{\text{Btu/hour}}{\text{ft}^2-\text{°F}}\end{array}\right),$$

or

$$Q = (T_{in} - T_{out}) \cdot A \cdot U.$$

In southwest Virginia, the number of heating days per year is approximately 230, and the annual heating degree-days equals 230 (65°F–46°F) = 4,370 degree-days per year. Here, 65°F is assumed to be the average inside temperature and 46°F is the average outside temperature each day.

Consider a 2,400-ft² single-story house in Blacksburg. The typical annual space-heating load for this size of a house is 100×10^6 Btu. That is, with no insulation in the attic, we lose about 100×10^6 Btu per year.* Common sense dictates that the "no insulation" alternative is not attractive and is to be avoided.

With insulation in the attic, the amount of heat lost each year will be reduced. The value of energy savings that results from adding insulation and reducing heat loss is dependent on what type of residential heating furnace is installed. For this example, we assume that an electrical resistance furnace is installed by the builder, and its efficiency is near 100%.

Now we're in a position to answer the following question: What amount of insulation is most economical? An additional piece of data we need involves the cost of electricity, which is \$0.074 per kWh. This can be converted to dollars per 10^6 Btu as follows (1 kWh = 3,413 Btu):

$$\frac{\text{kWh}}{3{,}413 \text{ Btu}} = 293 \text{ kWh per million Btu}$$

* 100×10^6 Btu/yr $\cong \left(\frac{4{,}370\ \text{°F-days per year}}{1.00\ \text{efficiency}}\right)(2{,}400\ \text{ft}^2)(24\ \text{hours/day})\left(\frac{0.397\ \text{Btu/hr}}{\text{ft}^2\text{-°F}}\right)$, where 0.397 is the U-factor with no insulation.

$$\frac{293 \text{ kWh}}{10^6 \text{ Btu}}\left(\frac{\$0.074}{\text{kWh}}\right) \cong \$21.75/10^6 \text{ Btu}.$$

The cost of several insulation alternatives and associated space-heating loads for this house are given in the following table:

	Amount of Insulation			
	R11	*R19*	*R30*	*R38*
Investment cost (\$)	600	900	1,300	1,600
Annual heating load (Btu/year)	74×10^6	69.8×10^6	67.2×10^6	66.2×10^6

In view of these data, which amount of attic insulation is most economical? The life of the insulation is estimated to be 25 years.

Solution

Set up a table to examine total life-cycle costs:

	R11	*R19*	*R30*	*R38*
A. Investment cost	\$600	\$900	\$1,300	\$1,600
B. Cost of heat loss per year	\$1,609.50	\$1,518.15	\$1,461.60	\$1,439.85
C. Cost of heat loss over 25 years	\$40,237.50	\$37,953.75	\$36,540	\$35,996.25
D. Total life cycle costs (A + C)	\$40,837.50	\$38,853.75	\$37,840	\$37,596.25

Answer: To minimize total life-cycle costs, select R38 insulation.

Caution

This conclusion may change when we consider the time value of money (i.e., an interest rate greater than zero) in Chapter 4. In such a case, it will not necessarily be true that adding more and more insulation is the optimal course of action.

2.4 Present Economy Studies

When alternatives for accomplishing a specific task are being compared over *one year or less* and the influence of time on money can be ignored, engineering economic analyses are referred to as *present economy studies*. Several situations involving present economy studies are illustrated in this section. The rules, or criteria, shown next will be used to select the preferred alternative when defect-free output (yield) is variable *or* constant among the alternatives being considered.

RULE 1: When revenues and other economic benefits are present and vary among alternatives, choose the alternative that *maximizes* overall profitability based on the number of defect-free units of a product or service produced.

RULE 2: When revenues and other economic benefits are *not* present *or* are constant among all alternatives, consider only the costs and select the alternative that *minimizes* total cost per defect-free unit of product or service output.

2.4.1 Total Cost in Material Selection

In many cases, economic selection among materials cannot be based solely on the costs of materials. Frequently, a change in materials will affect the design and processing costs, and shipping costs may also be altered.

EXAMPLE 2-8 **Choosing the Most Economic Material for a Part**

A good example of this situation is illustrated by a part for which annual demand is 100,000 units. The part is produced on a high-speed turret lathe, using 1112 screw-machine steel costing $0.30 per pound. A study was conducted to determine whether it might be cheaper to use brass screw stock, costing $1.40 per pound. Because the weight of steel required per piece was 0.0353 pounds and that of brass was 0.0384 pounds, the material cost per piece was $0.0106 for steel and $0.0538 for brass. However, when the manufacturing engineering department was consulted, it was found that, although 57.1 defect-free parts per hour were being produced by using steel, the output would be 102.9 defect-free parts per hour if brass were used. Which material should be used for this part?

Solution

The machine attendant was paid $15.00 per hour, and the variable (i.e., traceable) overhead costs for the turret lathe were estimated to be $10.00 per hour. Thus, the total cost comparison for the two materials was as follows:

	1112 Steel	Brass
Material	$0.30 × 0.0353 = $0.0106	$1.40 × 0.0384 = $0.0538
Labor	$15.00/57.1 = 0.2627	$15.00/102.9 = 0.1458
Variable overhead	$10.00/57.1 = 0.1751	$10.00/102.9 = 0.0972
Total cost per piece	$0.4484	$0.2968
Saving per piece by use of brass = $0.4484 − $0.2968 = $0.1516		

Because 100,000 parts are made each year, revenues are constant across the alternatives. Rule 2 would select brass, and its use will produce a savings of $151.60 per thousand (a total of $15,160 for the year). It is also clear that costs other than the cost of material were important in the study.

Care should be taken in making economic selections between materials to ensure that any differences in shipping costs, yields, or resulting scrap are taken into account. Commonly, alternative materials do not come in the same stock sizes, such as sheet sizes and bar lengths. This may considerably affect the yield obtained from a given weight of material. Similarly, the resulting scrap may differ for various materials.

In addition to deciding what material a product should be made of, there are often alternative methods or machines that can be used to produce the product, which, in turn, can impact processing costs. Processing times may vary with the machine selected, as may the product yield. As illustrated in Example 2-9, these considerations can have important economic implications.

EXAMPLE 2-9 **Choosing the Most Economical Machine for Production**

Two currently owned machines are being considered for the production of a part. The capital investment associated with the machines is about the same and can be ignored for purposes of this example. The important differences between the machines are their production capacities (production rate × available production hours) and their reject rates (percentage of parts produced that cannot be sold). Consider the following table:

	Machine *A*	Machine *B*
Production rate	100 parts/hour	130 parts/hour
Hours available for production	7 hours/day	6 hours/day
Percent parts rejected	3%	10%

The material cost is $6.00 per part, and all defect-free parts produced can be sold for $12 each. (Rejected parts have negligible scrap value.) For either machine, the operator cost is $15.00 per hour and the variable overhead rate for traceable costs is $5.00 per hour.

(a) Assume that the daily demand for this part is large enough that all defect-free parts can be sold. Which machine should be selected?

(b) What would the percent of parts rejected have to be for Machine *B* to be as profitable as Machine *A*?

Solution

(a) Rule 1 applies in this situation because total daily revenues (selling price per part times the number of parts sold per day) and total daily costs will vary depending on the machine chosen. Therefore, we should select the machine that will maximize the profit per day:

$$\begin{aligned}\text{Profit per day} &= \text{Revenue per day} - \text{Cost per day}\\ &= (\text{Production rate})(\text{Production hours})(\$12/\text{part})\\ &\quad\times[1-(\%\text{rejected}/100)]\\ &\quad-(\text{Production rate})(\text{Production hours})(\$6/\text{part})\\ &\quad-(\text{Production hours})(\$15/\text{hour}+\$5/\text{hour}).\end{aligned}$$

$$\begin{aligned}\text{Machine } A\text{: Profit per day} &= \left(\frac{100 \text{ parts}}{\text{hour}}\right)\left(\frac{7 \text{ hours}}{\text{day}}\right)\left(\frac{\$12}{\text{part}}\right)(1-0.03)\\ &\quad-\left(\frac{100 \text{ parts}}{\text{hour}}\right)\left(\frac{7 \text{ hours}}{\text{day}}\right)\left(\frac{\$6}{\text{part}}\right)\\ &\quad-\left(\frac{7 \text{ hours}}{\text{day}}\right)\left(\frac{\$15}{\text{hour}}+\frac{\$5}{\text{hour}}\right)\\ &= \$3{,}808 \text{ per day.}\end{aligned}$$

$$\begin{aligned}\text{Machine } B\text{: Profit per day} &= \left(\frac{130 \text{ parts}}{\text{hour}}\right)\left(\frac{6 \text{ hours}}{\text{day}}\right)\left(\frac{\$12}{\text{part}}\right)(1-0.10)\\ &\quad-\left(\frac{130 \text{ parts}}{\text{hour}}\right)\left(\frac{6 \text{ hours}}{\text{day}}\right)\left(\frac{\$6}{\text{part}}\right)\\ &\quad-\left(\frac{6 \text{ hours}}{\text{day}}\right)\left(\frac{\$15}{\text{hour}}+\frac{\$5}{\text{hour}}\right)\\ &= \$3{,}624 \text{ per day.}\end{aligned}$$

Therefore, *select Machine A* to maximize profit per day.

(b) To find the breakeven percent of parts rejected, X, for Machine B, set the profit per day of Machine A equal to the profit per day of Machine B, and solve for X:

$$\begin{aligned}\$3{,}808/\text{day} &= \left(\frac{130 \text{ parts}}{\text{hour}}\right)\left(\frac{6 \text{ hours}}{\text{day}}\right)\left(\frac{\$12}{\text{part}}\right)(1-X)-\left(\frac{130 \text{ parts}}{\text{hour}}\right)\\ &\quad\times\left(\frac{6 \text{ hours}}{\text{day}}\right)\left(\frac{\$6}{\text{part}}\right)-\left(\frac{6 \text{ hours}}{\text{day}}\right)\left(\frac{\$15}{\text{hour}}+\frac{\$5}{\text{hour}}\right).\end{aligned}$$

Thus, X = 0.08, so the percent of parts rejected for Machine B can be no higher than 8% for it to be as profitable as Machine A.

2.4.2 Making versus Purchasing (Outsourcing) Studies*

In the short run, say, one year or less, a company may consider producing an item in-house even though the item can be purchased (outsourced) from a supplier at a price lower than the company's standard production costs. (See Section 2.1.2.) This could occur if (1) direct, indirect, and overhead costs are incurred regardless of whether the item is purchased from an outside supplier and (2) the *incremental* cost of producing an item in the short run is less than the supplier's price. Therefore, the relevant short-run costs of make versus purchase decisions are the *incremental costs* incurred and the *opportunity costs* of the resources involved.

Opportunity costs may become significant when in-house manufacture of an item causes other production opportunities to be forgone (often because of insufficient capacity). But in the long run, capital investments in additional manufacturing plant and capacity are often feasible alternatives to outsourcing. (Much of this book is concerned with evaluating the economic worthiness of proposed capital investments.) Because engineering economy often deals with *changes* to existing operations, standard costs may not be too useful in make-versus-purchase studies. In fact, if they are used, standard costs can lead to uneconomical decisions. Example 2-10 illustrates the correct procedure to follow in performing make-versus-purchase studies based on incremental costs.

EXAMPLE 2-10 **To Produce or Not to Produce?—That Is the Question**

A manufacturing plant consists of three departments: A, B, and C. Department A occupies 100 square meters in one corner of the plant. Product X is one of several products being produced in Department A. The daily production of Product X is 576 pieces. The cost accounting records show the following average daily production costs for Product X:

Direct labor	(1 operator working 4 hours per day at \$22.50/hr, including fringe benefits, plus a part-time foreman at \$30 per day)	\$120.00
Direct material		86.40
Overhead	(at \$0.82 per square meter of floor area)	82.00
	Total cost per day =	\$288.40

The department foreman has recently learned about an outside company that sells Product X at \$0.35 per piece. Accordingly, the foreman figured a cost per day of \$0.35(576) = \$201.60, resulting in a daily savings of \$288.40−\$201.60 = \$86.80. Therefore, a proposal was submitted to the plant manager for shutting down the production line of Product X and buying it from the outside company.

* Much interest has been shown in outsourcing decisions. For example, see P. Chalos, "Costing, Control, and Strategic Analysis in Outsourcing Decisions," *Journal of Cost Management*, 8, no. 4 (Winter 1995): pp. 31–37.

However, after examining each component separately, the plant manager decided not to accept the foreman's proposal based on the unit cost of Product X:

1. *Direct labor:* Because the foreman was supervising the manufacture of other products in Department *A* in addition to Product X, the only possible savings in labor would occur if the operator working 4 hours per day on Product X were not reassigned after this line is shut down. That is, a maximum savings of $90.00 per day would result.
2. *Materials:* The maximum savings on direct material will be $86.40. However, this figure could be lower if some of the material for Product X is obtained from scrap of another product.
3. *Overhead:* Because other products are made in Department *A*, no reduction in total floor space requirements will probably occur. Therefore, no reduction in overhead costs will result from discontinuing Product X. It has been estimated that there will be daily savings in the variable overhead costs traceable to Product X of about $3.00 due to a reduction in power costs and in insurance premiums.

Solution

If the manufacture of Product X is discontinued, the firm will save at most $90.00 in direct labor, $86.40 in direct materials, and $3.00 in variable overhead costs, which totals $179.40 per day. This estimate of actual cost savings per day is less than the potential savings indicated by the cost accounting records ($288.40 per day), and it would not exceed the $201.60 to be paid to the outside company if Product X is purchased. For this reason, the plant manager used Rule 2 and rejected the proposal of the foreman and continued the manufacture of Product X.

In conclusion, Example 2-10 shows how an erroneous decision might be made by using the unit cost of Product X from the cost accounting records without detailed analysis. The fixed cost portion of Product X's unit cost, which is present even if the manufacture of Product X is discontinued, was not properly accounted for in the original analysis by the foreman.

2.4.3 Trade-Offs in Energy Efficiency Studies

Energy efficiency affects the annual expense of operating an electrical device such as a pump or motor. Typically, a more energy-efficient device requires a higher capital investment than does a less energy-efficient device, but the extra capital investment usually produces annual savings in electrical power expenses relative to a second pump or motor that is less energy efficient. This important trade-off between capital investment and annual electric power consumption will be considered in several chapters of this book. Hence, the purpose of Section 2.4.3 is to explain how the annual expense of operating an electrical device is calculated and traded off against capital investment cost.

If an electric pump, for example, can deliver a given horsepower (hp) or kilowatt (kW) rating to an industrial application, the *input* energy requirement is determined by dividing the given output by the energy efficiency of the device. The input requirement in hp or kW is then multiplied by the annual hours that the device operates and the unit cost of electric power. You can see that the higher the efficiency of the pump, the lower the annual cost of operating the device is relative to another less-efficient pump.

EXAMPLE 2-11

Investing in Electrical Efficiency

Two pumps capable of delivering 100 hp to an agricultural application are being evaluated in a present economy study. The selected pump will only be utilized for one year, and it will have no market value at the end of the year. Pertinent data are summarized as follows:

	ABC Pump	XYZ Pump
Purchase price	$2,900	$6,200
Annual maintenance	$170	$510
Efficiency	80%	90%

If electric power costs $0.10 per kWh and the pump will be operated 4,000 hours per year, which pump should be chosen? Recall that 1 hp = 0.746 kW.

Solution

The annual expense of electric power for the ABC pump is

$$(100 \text{ hp}/0.80)(0.746 \text{ kW/hp})(\$0.10/\text{kWh})(4{,}000 \text{ hours/yr}) = \$37{,}300.$$

For the XYZ Pump, the annual expense of electric power is

$$(100 \text{ hp}/0.90)(0.746 \text{ kW/hp})(\$0.10/\text{kWh})(4{,}000 \text{ hours/yr}) = \$33{,}156.$$

Thus, the total annual cost of owning and operating the ABC pump is $40,370, while the total cost of owning and operating the XYZ pump for one year is $39,866. Consequently, the more energy-efficient XYZ pump should be selected to minimize total annual cost. Notice the difference in annual energy expense ($4,144) that results from a 90% efficient pump relative to an 80% efficient pump. This cost reduction more than balances the extra $3,300 in capital investment and $340 in annual maintenance required for the XYZ pump.

2.5 CASE STUDY—The Economics of Daytime Running Lights

The use of Daytime Running Lights (DRLs) has increased in popularity with car designers throughout the world. In some countries, motorists are required to drive with their headlights *on* at all times. U.S. car manufacturers now offer models equipped with daytime running lights. Most people would agree that driving with the headlights on at night is cost effective with respect to extra *fuel* consumption and safety considerations (not to mention required by law!). Cost effective means that benefits outweigh (exceed) the costs. However, some consumers have questioned whether it is cost effective to drive with your headlights on during the day.

In an attempt to provide an answer to this question, let us consider the following suggested data:

75% of driving takes place during the daytime.
2% of fuel consumption is due to accessories (radio, headlights, etc.).
Cost of fuel = \$3.00 per gallon.
Average distance traveled per year = 15,000 miles.
Average cost of an accident = \$2,800.
Purchase price of headlights = \$25.00 per set (2 headlights).
Average time car is in operation per year = 350 hours.
Average life of a headlight = 200 operating hours.
Average fuel consumption = 1 gallon per 30 miles.

Let's analyze the cost effectiveness of driving with headlights on during the day by considering the following set of questions:

- What are the extra costs associated with driving with headlights on during the day?
- What are the benefits associated with driving with headlights on during the day?
- What additional assumptions (if any) are needed to complete the analysis?
- Is it cost effective to drive with headlights on during the day?

Solution

After some reflection on the above questions, you could reasonably contend that the extra costs of driving with headlights on during the day include increased fuel consumption and more frequent headlight replacement. Headlights increase visibility to other drivers on the road. Another possible benefit is the reduced chance of an accident.

Additional assumptions needed to consider during our analysis of the situation include:

1. the percentage of fuel consumption due to headlights alone and
2. how many accidents can be avoided per unit time.

Selecting the dollar as our common unit of measure, we can compute the extra cost associated with daytime use of headlights and compare it to the expected benefit (also measured in dollars). As previously determined, the extra costs include increased fuel consumption and more frequent headlight replacement.

Let's develop an estimate of the annual fuel cost:

$$\text{Annual fuel cost} = (15{,}000 \text{ mi/yr})(1 \text{ gal}/30 \text{ mi})(\$3.00/\text{gal}) = \$1{,}500/\text{yr}.$$

Assume (worst case) that 2% of fuel consumption is due to normal (night-time) use of headlights.

$$\text{Fuel cost due to normal use of headlights} = (\$1{,}500/\text{yr})(0.02) = \$30/\text{yr}.$$

$$\text{Fuel cost due to continuous use of headlights} = (4)(\$30/\text{yr}) = \$120/\text{yr}.$$

$$\text{Headlight cost for normal use} = (0.25)\left(\frac{350 \text{ hours/yr}}{200 \text{ hours/set}}\right)\left(\frac{\$25}{\text{set}}\right) = \$10.94/\text{yr}.$$

$$\text{Headlight cost for continuous use} = \left(\frac{350 \text{ hours/yr}}{200 \text{ hours/set}}\right)\left(\frac{\$25}{\text{set}}\right) = \$43.75/\text{yr}.$$

$$\begin{aligned}\text{Total cost associated with daytime use} &= (\$120 - \$30) + (\$43.75 - \$10.94)\\ &= \$122.81/\text{yr}.\end{aligned}$$

If driving with your headlights on during the day results in at least one accident being avoided during the next (\$2,800)/(\$122.81) = 22.8 years, then the continuous use of your headlights is cost effective. Although in the short term, you may be able to contend that the use of DRLs lead to increased fuel and replacement bulb costs, the benefits of increased personal safety and mitigation of possible accident costs in the long run more than offset the apparent short-term cost savings.

2.6 Summary

In this chapter, we have discussed cost terminology and concepts important in engineering economy. It is important that the meaning and use of various cost terms and concepts be understood in order to communicate effectively with other engineering and management personnel. A listing of important abbreviations and notation, by chapter, is provided in Appendix B.

Several general economic concepts were discussed and illustrated. First, the ideas of consumer and producer goods and services, measures of economic growth, competition, and necessities and luxuries were covered. Then, some relationships among costs, price, and volume (demand) were discussed. Included were the concepts of optimal volume and breakeven points. Important economic principles of design optimization were also illustrated in this chapter.

The use of present-economy studies in engineering decision making can provide satisfactory results and save considerable analysis effort. When an adequate engineering economic analysis can be accomplished by considering the various monetary consequences that occur in a short time period (usually one year or less), a present-economy study should be used.

Problems

The number in parentheses that follows each problem refers to the section from which the problem is taken.

2-1. A company in the process industry produces a chemical compound that is sold to manufacturers for use in the production of certain plastic products. The plant that produces the compound employs approximately 300 people. Develop a list of six different cost elements that would be *fixed* and a similar list of six cost elements that would be *variable*. **(2.1)**

2-2. Classify each of the following cost items as mostly fixed or variable: **(2.1)**

Raw materials	Administrative salaries
Direct labor	Payroll taxes
Depreciation	Insurance (building and equipment)
Supplies	Clerical salaries
Utilities	Sales commissions
Property taxes	Rent
Interest on borrowed money	

2-3. A team of researchers has developed a process for extracting combustible methane gas from garbage. With a specially adapted internal combustion engine, the team claims that an automobile can be propelled 25 km per day from the garbage accumulated by a small village. Their experimental car travels 85 km per day for an estimated cost of $5 (this is the allocated cost for methane process equipment—the garbage is essentially free). **(2.1)**

a. How many similarly populated villages are required to produce the garbage to fuel 150,000 km of annual driving by a fleet of cars? What is the annual cost?

b. How does your answer to Part (a) compare to a gasoline-fueled car averaging 15 km per liter when the cost of gasoline is $2.00 per liter?

2-4. John Bros, owner of a spark plug manufacturing facility, is looking to expand his production capacity. He is considering three locations A, B, and C for the construction of a new plant. The company wishes to find the most economical location for an expected volume of 2,500 units per year. Bros calculates that the fixed costs per year at each of the sites amount to $25,000, $50,000, and $100,000 respectively. The variable cost is $70 per unit, $40 per unit and $20 per unit respectively. The expected selling price for each spark plug is $120. Find the preference volume for each location. **(2.1)**

2-5. Stan Moneymaker presently owns a 10-year-old automobile with low mileage (78,000 miles). The NADA "blue book" value of the car is $2,500. Unfortunately, the car's transmission just failed, and Stan decided to spend $1,500 to have it repaired. Now, six months later, Stan has decided to sell the car, and he reasons that his asking price should be $2,500 + $1,500 = $4,000. Comment on the wisdom of Stan's logic. If he receives an offer for $3,000, should he accept it? Explain your reasoning. **(2.1)**

2-6. You have been invited by friends to fly to Germany for Octoberfest next year. For international travel, you apply for a passport that costs $97 and is valid for 10 years. After you receive your passport, your travel companions decide to cancel the trip because of "insufficient funds." You decide to also cancel your travel plans because traveling alone is no fun. Is your passport expense a sunk cost or an opportunity cost? Explain your answer. **(2.1)**

2-7. The fixed and variable costs for three manufacturing plant sites for a product are shown in the following table:

Site	Fixed Cost per Year	Variable Cost per Unit
A	$500	$10
B	$1,000	$8
C	$1,500	$6

a. Over what range of production is each location optimal?

b. Which site is best for the production of 200 units? **(2.1)**

2-8. A friend of yours has been thinking about quitting her regular day job and going into business for herself. She currently makes $60,000 per year as an employee of the Ajax Company, and she anticipates no raise for at least another year. She believes she can make $200,000 as an independent consultant in six-sigma "black belt" training for large corporations. Her start-up expenses are expected to be $120,000 over the next year. If she decides to keep her current job, what is the expected opportunity cost of this decision? Attempt to balance the pros and cons of the option that your friend is turning away from. **(2.1)**

2-9. Duncan Jones has made a windfall gain of $50,000. He has come up with three options of investing the money. First, he could dispose of his bachelor pad and buy a house in a nice neighbourhood. Second, he could invest the money in carefully chosen stocks and shares, which are expected to increase in value by 30% per year. This, however, is a risky option. Third, he could put his money into a fixed deposit arrangement with a bank and earn 8.5% per year. There is little risk involved in taking the third option. **(2.1)**

a. If he decides to purchase a house, what is the opportunity cost of this choice? Explain your reasoning.

b. If he invests in the stock market, what is the opportunity cost of this choice? Explain your reasoning.

2-10. In your own words, describe the life-cycle cost concept. Why is the potential for achieving life-cycle cost savings greatest in the acquisition phase of the life cycle? **(2.1)**

2-11. A big manufacturing concern runs a number of machine tools, each with a maximum production capacity of 150 items per day. The management of the company claims that the economic breakeven point with these machines is 70 items per day. **(2.2)**

a. Draw a conceptual graph to show the total revenue and total costs that this company is experiencing.

b. Identify all types of fixed costs that the manufacturing concern should carefully examine to lower its breakeven point. Explain your reasoning.

c. Identify various variable costs that could possibly be reduced to lower the breakeven point. Why did you select these cost items?

2-12. A company manufactures microprocessors for computers whose cost function is given by $C = 6X + 9$, where X is the number of microprocessors. The selling price per unit is $P = 30 - 3X$ and maximum output of the plant is 5000 units per month. **(2.2)**

a. Determine the optimum demand for this product.

b. What is the maximum profit per month?

c. At what volume does breakeven occur?

d. What is the company's range of profitable demand?

2-13. A firm operates in a perfectly competitive market whose total cost varies as $TC = X^3 - 3X^2 - 10X + 2$, and the price of the product they manufacture is given by $P = 130 - 2X$, where X is number of units of output. **(2.2)**

a. What optimum number of units should be produced in order to maximize profits?

b. What is the profit if the optimum number of units is produced?

c. What is the number of units that should be produced and sold each month in order to maximize profits?

d. Show that your answer to Part (a) maximizes profit.

2-14. A large wood products company is negotiating a contract to sell plywood overseas. The fixed cost that can be allocated to the production of plywood is $900,000 per month. The variable cost per thousand board feet is $131.50. The price charged will be determined by $p = \$600 - (0.05)D$ per 1,000 board feet. **(2.2)**

a. For this situation, determine the optimal monthly sales volume for this product and calculate the profit (or loss) at the optimal volume.

b. What is domain of profitable demand during a month?

2-15. A company produces and sells a consumer product and is able to control the demand for the product by varying the selling price. The approximate relationship between price and demand is

$$p = \$38 + \frac{2{,}700}{D} - \frac{5{,}000}{D^2}, \text{ for } D > 1,$$

where p is the price per unit in dollars and D is the demand per month. The company is seeking to maximize its profit. The fixed cost is $1,000 per month and the variable cost (c_v) is $40 per unit. **(2.2)**

a. What is the number of units that should be produced and sold each month to maximize profit?

b. Show that your answer to Part (a) maximizes profit.

2-16. An electric power plant uses solid waste for fuel in the production of electricity. The cost Y in dollars per hour to produce electricity is $Y = 12 + 0.3X + 0.27X^2$, where X is in megawatts. Revenue in dollars per hour from the sale of electricity is $15X - 0.2X^2$. Find the value of X that gives maximum profit. **(2.2)**

2-17. A soft drink company with bottling plant automation software installed incurs an annual fixed cost of \$10,000. Direct labor is \$3.50 per package and material cost is \$4.50 per package. The selling price will be \$12.50 per package. What is the breakeven point S in \$ and in units. **(2.2)**

2-18. A manufacturing concern produces a product which is sold at a price of \$10.50 per unit. The plant's fixed cost is \$50,000 and its variable cost is \$6.50 per unit. How many units should be produced at the breakeven point? How many units must be produced in order to earn a profit of \$10,000? What would the profit on a sales volume of 20,000 units be? **(2.2)**

2-19. A company, XYZ has an estimated sales volume of 200 units with unit sales price of \$25. Compute the breakeven point using the information given in the following table, if: **(2.2)**

a. Fixed cost increases by 10%

b. Variable cost increases by 10%

c. Sales price increases by 10%

Item	Fixed Cost (\$)	Variable Cost (\$)
Direct material cost	-	900
Direct labor cost	-	1,000
Overheads	2,000	900

2-20. A plant produces a commodity whose selling price is given by the function $P = \$500 - 0.0025D$ per unit, where D is the number of units manufactured. Total fixed cost is \$26,335 per year and variable cost $V = \$440 - 0.005D$. The plant is designed to manufacture 5000 units per year. Determine the level of output that produces the maximum annual profit and breakeven point. On a graph, show the curves for average variable production cost per unit, incremental cost per unit and incremental annual profit from sales per unit. **(2.3)**

2-21. A regional airline company estimated four years ago that each pound of aircraft weight adds \$30 per year to its fuel expense. Now the cost of jet fuel has doubled from what it was four years ago. A recent engineering graduate employed by the company has made a recommendation to reduce fuel consumption of an aircraft by installing leather seats as part of a "cabin refurbishment program." The total reduction in weight would be approximately 600 pounds per aircraft. If seats are replaced annually (a worst-case situation), how much can this airline afford to spend on the cabin refurbishments? What nonmonetary advantages might be associated with the refurbishments? Would you support the engineer's recommendation? **(2.1)**

2-22. Jerry Smith's residential air conditioning (AC) system has not been able to keep his house cool enough in 90°F weather. He called his local AC maintenance person, who discovered a leak in the evaporator. The cost to recharge the AC unit is \$40 for gas and \$45 for labor, but the leak will continue and perhaps grow worse. The AC person cautioned that this service would have to be repeated each year unless the evaporator is replaced. A new evaporator would run about \$800–\$850.

Jerry reasons that fixing the leak in the evaporator on an annual basis is the way to go. "After all, it will take 10 years of leak repairs to equal the evaporator's replacement cost." Comment on Jerry's logic. What would you do? **(2.1)**

2-23. Ethanol blended with gasoline can be used to power a "flex-fueled" car. One particular blend that is gaining in popularity is E85, which is 85% ethanol and 15% gasoline. E85 is 80% cleaner burning than gasoline alone, and it reduces our dependency on foreign oil. But a flex-fueled car costs \$1,000 more than a conventional gasoline-fueled car. Additionally, E85 fuel gets 10% less miles per gallon than a conventional automobile.

Consider a 100% gasoline-fueled car that averages 30 miles per gallon. The E85-fueled car will average about 27 miles per gallon. If either car will be driven 81,000 miles before being traded in, how much will the E85 fuel have to cost (per gallon) to make the flex-fueled car as economically attractive as a conventional gasoline-fueled car? Gasoline costs \$2.89 per gallon. Work this problem without considering the time value of money. **(2.1)**

2-24. The fixed cost for a steam line per meter of pipe is $\$450X + \$50$ per year. The cost for loss of heat from the pipe per meter is $\$4.8/X^{1/2}$ per year. Here, X represents the thickness of insulation in meters, and X is a continuous design variable. **(2.3)**

a. What is the optimum thickness of the insulation?

b. How do you know that your answer in Part (a) minimizes total cost per year?

c. What is the basic trade-off being made in this problem?

2-25. A farmer estimates that if he harvests his soybean crop now, he will obtain 1,000 bushels, which he can sell at \$3.00 per bushel. However, he estimates that this crop will increase by an additional 1,200 bushels of soybeans for each week he delays harvesting, but the price will drop at a rate of 50 cents per bushel per week; in addition, it is likely that he will experience spoilage of approximately 200 bushels per week for each week he delays harvesting. When should he harvest his crop to obtain the largest net cash return, and how much will be received for his crop at that time? **(2.3)**

2-26. The cost of operating a large ship (C_O) varies as the square of its velocity (v); specifically, $C_O = knv^2$, where n is the trip length in miles and k is a constant of proportionality. It is known that at 12 miles/hour, the *average* cost of operation is \$100 per mile. The owner of the ship wants to minimize the cost of operation, but it must be balanced against the cost of the perishable cargo (C_c), which the customer has set at \$1,500 per hour. At what velocity should the trip be planned to minimize the total cost (C_T), which is the sum of the cost of operating the ship and the cost of perishable cargo? **(2.3)**

2-27. A manufacturing concern's total cost function is $C = X/5$ and its revenue function is $R = 39\log(8X + 1)$. Find the optimum output where the profit is maximized and the maximum profit. **(2.3)**

2-28. According to the U.S. Department of the Interior, the amount of energy lost because of poorly insulated homes is equivalent to 2 million barrels of oil per day. In 2009, this is more oil than the United States imports from Saudi Arabia each day. If we were to insulate our homes as determined by Example 2-7, we could eliminate our oil dependence on Saudi Arabia. If the cost of electricity increases to \$0.15 per kWh and the cost of insulation quadruples, how much insulation should be chosen in Example 2-7? **(2.3)**

2-29. One component of a system's life-cycle cost is the cost of system failure. Failure costs can be reduced by designing a more reliable system. A simplified expression for system life-cycle cost, C, can be written as a function of the system's failure rate:

$$C = \frac{C_I}{\lambda} + C_R \cdot \lambda \cdot t.$$

Here, C_I = investment cost (\$ per hour per failure),
C_R = system repair cost,
λ = system failure rate (failures/operating hour),
t = operating hours.

a. Assume that C_I, C_R, and t are constants. Derive an expression for λ, say λ^*, that optimizes C. **(2.3)**

b. Does the equation in Part (a) correspond to a maximum or minimum value of C? Show all work to support your answer.

c. What trade-off is being made in this problem?

2-30. Stan Moneymaker has been shopping for a new car. He is interested in a certain 4-cylinder sedan that averages 28 miles per gallon. But the salesperson tried to persuade Stan that the 6-cylinder model of the same automobile only costs \$2,500 more and is really a "more sporty and responsive" vehicle. Stan is impressed with the zip of the 6-cylinder car and reasons that \$2,500 is not too much to pay for the extra power.

How much extra is Stan really paying if the 6-cylinder car averages 22 miles per gallon? Assume that Stan will drive either automobile 100,000 miles, gasoline will average \$3.00 per gallon, and maintenance is roughly the same for both cars. State other assumptions you think are appropriate. **(2.4)**

2-31. A producer of synthetic motor oil for automobiles and light trucks has made the following statement: "One quart of Dynolube added to your next oil change will increase fuel mileage by one percent. This one-time additive will improve your fuel mileage over 50,000 miles of driving." **(2.4)**

a. Assume the company's claim is correct. How much money will be saved by adding one quart of Dynolube if gasoline costs \$3.00 per gallon and your car averages 20 miles per gallon without the Dynolube?

b. If a quart of Dynolube sells for \$19.95, would you use this product in your automobile?

2-32. An automobile dealership offers to fill the four tires of your new car with 100% nitrogen for a cost of \$20. The dealership claims that nitrogen-filled tires run cooler than those filled with compressed air, and they advertise that nitrogen extends tire mileage (life) by 25%. If new tires cost \$50 each and are guaranteed to get 50,000 miles (filled with air) before they require replacement, is the dealership's offer a good deal? **(2.4)**

2-33. In the design of an automobile radiator, an engineer has a choice of using either a brass–copper alloy casting or a plastic molding. Either material provides the same service. However, the brass–copper alloy casting weighs 25 pounds, compared with

Albany, NY. For transportation, the group will rent a car from either the State Tech Motor Pool or a local car dealer. The Motor Pool charges \$0.36 per mile, has no daily fee, and pays for the gas. The local car dealer charges \$30 per day and \$0.20 per mile, but the group must pay for the gas. The car's fuel rating is 20 miles per gallon, and the price of gas is estimated to be \$2.00 per gallon. **(2.2)**

a. At what point, in miles, is the cost of both options equal?
b. The car dealer has offered a special student discount and will give the students 100 free miles per day. What is the new breakeven point?
c. Suppose now that the Motor Pool reduces its all-inclusive rate to \$0.34 per mile and that the car dealer increases the rate to \$30 per day and \$0.28 per mile. In this case, the car dealer wants to encourage student business, so he offers 900 free miles for the entire six-day trip. He claims that if more than 750 miles are driven, students will come out ahead with one of his rental cars. If the students anticipate driving 2,000 miles (total), from whom should they rent a car? Is the car dealer's claim entirely correct?

2-47. *Web Exercise* Home heating accounts for approximately one-third of energy consumption in a typical U.S. household. Despite soaring prices of oil, coal, and natural gas, one can make his/her winter heating bill noninflationary by installing an ultraconvenient corn burning stove that costs in the neighborhood of \$2,400. That's right—a small radiant-heating stove that burns corn and adds practically nothing to global warming or air pollution can be obtained through www.magnumfireplace.com. Its estimated annual savings per household in fuel is \$300 in a regular U.S. farming community.

Conduct research on this means of home heating by accessing the above Web site. Do the annual savings you determine in your locale for a 2,000-square foot ranch-style house more than offset the cost of installing and maintaining a corn-burning stove? What other factors besides dollars might influence your decision to use corn for your home heating requirements? Be specific with your suggestions. **(2.4)**

Spreadsheet Exercises

2-48. Refer to Example 2-4. If your focus was on reducing expenses, would it be better to reduce the fixed cost (B1) or variable cost (B2) component? What is the effect of a ± 10% change in both of these factors? **(2.2)**

2-49. Refer to Example 2-7. If the average inside temperature of this house in Virginia is increased from 65°F to 72°F, what is the most economical insulation amount? Assume that 100,000,000 Btu are lost with no insulation when the thermostat is set at 65°F. The cost of electricity is now \$0.086 per kWh. In addition, the cost of insulation has increased by 50%. Develop a spreadsheet to solve this problem. **(2.3)**

Case Study Exercises

2-50. What are the key factors in this analysis, and how would your decision change if the assumed value of these factors changes? For example, what impact does rising fuel costs have on this analysis? Or, what if studies have shown that drivers can expect to avoid at least one accident every 10 years due to daytime use of headlights? **(2.5)**

2-51. Visit your local car dealer (either in person or online) to determine the cost of the daytime running lights option. How many accidents (per unit time) would have to be avoided for this option to be cost effective? **(2.5)**

FE Practice Problems

A company has determined that the price and the monthly demand of one of its products are related by the equation

$$D = \sqrt{(400 - p)},$$

where p is the price per unit in dollars and D is the monthly demand. The associated fixed costs are \$1,125/month, and the variable costs are \$100/unit. Use this information to answer Problems 2-52 and 2-53. Select the closest answer. **(2.2)**

2-52. What is the optimal number of units that should be produced and sold each month?

(a) 10 units (b) 15 units (c) 20 units
(d) 25 units

2-53. Which of the following values of D represents the breakeven point?

(a) 10 units (b) 15 units (c) 20 units
(d) 25 units

A manufacturing company leases a building for \$100,000 per year for its manufacturing facilities. In addition, the machinery in this building is being paid for in installments of \$20,000 per year. Each unit of the product produced costs \$15 in labor and \$10 in materials. The product can be sold for \$40. Use this information to answer Problems 2-54 through 2-56. Select the closest answer. **(2.2)**

2-54. How many units per year must be sold for the company to breakeven?

(a) 4,800 (b) 3,000 (c) 8,000
(d) 6,667 (e) 4,000

2-55. If 10,000 units per year are sold, what is the annual profit?

(a) \$280,000 (b) \$50,000 (c) \$150,000
(d) −\$50,000 (e) \$30,000

2-56. If the selling price is lowered to \$35 per unit, how many units must be sold each year for the company to earn a profit of \$60,000 per year?

(a) 12,000 (b) 10,000 (c) 16,000
(d) 18,000 (e) 5,143

2-57. The fixed costs incurred by a small genetics research lab are \$200,000 per year. Variable costs are 60% of the annual revenue. If annual revenue is \$300,000, the annual profit/loss is most nearly which answer below? **(2.2)**

(a) \$66,000 profit (b) \$66,000 loss
(c) \$80,000 profit (d) \$80,000 loss

2-58. A manufacturer makes 7,900,000 memory chips per year. Each chip takes 0.4 minutes of direct labor at the rate of \$8 per hour. The overhead costs are estimated at \$11 per direct labor hour. A new process will reduce the unit production time by 0.01 minutes. If the overhead cost will be reduced by \$5.50 for each hour by which total direct hours are reduced, what is the maximum amount you will pay for the new process? Assume that the new process must pay for itself by the end of the first year. **(2.4)**

(a) \$25,017 (b) \$1,066,500 (c) \$10,533
(d) \$17,775 (e) \$711,000

Appendix 2-A Accounting Fundamentals

Accounting is often referred to as the language of business. Engineers should make serious efforts to learn about a firm's accounting practice so that they can better communicate with top management. This section contains an extremely brief and simplified exposition of the elements of financial accounting in recording and summarizing transactions affecting the finances of the enterprise. These fundamentals apply to any entity (such as an individual or a corporation) called here a *firm*.

2-A.1 The Accounting Equation

All accounting is based on the *fundamental accounting equation*, which is

$$\text{Assets} = \text{liabilities} + \text{owners' equity}, \tag{2-A-1}$$

where *assets* are those things of monetary value that the firm possesses, *liabilities* are those things of monetary value that the firm owes, and *owners' equity* is the worth of what the firm owes to its stockholders (also referred to as *equities, net worth*, etc.). For example, typical accounts in each term of Equation (2-A-1) are as follows:

Asset Accounts =	Liability Accounts +	Owner's Equity Accounts
Cash	Short-term debt	Capital stock
Receivables	Payables	Retained earnings (income retained in the firm)
Inventories	Long-term debt	
Equipment		
Buildings		
Land		

The fundamental accounting equation defines the format of the *balance sheet*, which is one of the two most common accounting statements and which shows the financial position of the firm at any given point in time.

Another important, and rather obvious, accounting relationship is

$$\text{Revenues} - \text{expenses} = \text{profit (or loss)}. \qquad (2\text{-A-}2)$$

This relationship defines the format of the *income statement* (also commonly known as a *profit-and-loss statement*), which summarizes the revenue and expense results of operations *over a period of time*. Equation (2-A-1) can be expanded to take into account profit as defined in Equation (2-A-2):

$$\text{Assets} = \text{liabilities} + (\text{beginning owners' equity} + \text{revenue} - \text{expenses}). \qquad (2\text{-A-}3)$$

Profit is the increase in money value (not to be confused with cash) that results from a firm's operations and is available for distribution to stockholders. It therefore represents the return on owners' invested capital.

A useful analogy is that a balance sheet is like a snapshot of the firm at an instant in time, whereas an income statement is a summarized moving picture of the firm over an interval of time. It is also useful to note that revenue serves to increase owners' interests in a firm, but an expense serves to decrease the owners' equity amount for a firm.

To illustrate the workings of accounts in reflecting the decisions and actions of a firm, suppose that an individual decides to undertake an investment opportunity and the following sequence of events occurs over a period of one year:

1. Organize XYZ firm and invest \$3,000 cash as capital.
2. Purchase equipment for a total cost of \$2,000 by paying cash.
3. Borrow \$1,500 through a note to the bank.
4. Manufacture year's supply of inventory through the following:
   (a) Pay \$1,200 cash for labor.
   (b) Incur \$400 accounts payable for material.
   (c) Recognize the partial loss in value (depreciation) of the equipment amounting to \$500.
5. Sell on credit all goods produced for year, 1,000 units at \$3 each. Recognize that the accounting cost of these goods is \$2,100, resulting in an increase in equity (through profits) of \$900.
6. Collect \$2,200 of accounts receivable.
7. Pay \$300 of accounts payable and \$1,000 of bank note.

A simplified version of the accounting entries recording the same information in a format that reflects the effects on the fundamental accounting equation (with a "+" denoting an increase and a "−" denoting a decrease) is shown in Figure 2-A-1. A summary of the results is shown in Figure 2-A-2.

It should be noted that the profit for a period serves to increase the value of the owners' equity in the firm by that amount. Also, it is significant that the net cash flow from operation of \$700 (= \$2,200 − \$1,200 − \$300) is not the same as profit. This amount was recognized in transaction 4(c), in which capital consumption (depreciation) for equipment of \$500 was declared. Depreciation serves to convert part of an asset into an expense, which is then reflected in a firm's profits, as seen in Equation (2-A-2). Thus, the profit was \$900, or \$200 more than the net cash flow. For our purposes, revenue is recognized when it is earned, and expenses are recognized when they are incurred.

One important and potentially misleading indicator of after-the-fact financial performance that can be obtained from Figure 2-A-2 is "annual rate of return." If the invested capital is taken to be the owners' (equity) investment, the annual rate of return at the end of this particular year is \$900/\$3,900 = 23%.

Financial statements are usually most meaningful if figures are shown for two or more years (or other reporting periods such as quarters or months) or for two or more individuals or firms. Such comparative figures can be used to reflect trends or financial indications that are useful in enabling investors and management to determine the effectiveness of investments *after* they have been made.

2-A.2 Cost Accounting

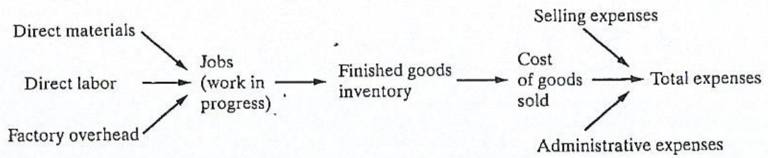
Cost accounting, or management accounting, is a phase of accounting that is of particular importance in engineering economic analysis because it is concerned principally with decision making and control in a firm. Consequently, cost accounting is the source of much of the cost data needed in making engineering economy studies. Modern cost accounting may satisfy any or all of the following objectives:

1. Determination of the actual cost of products or services
2. Provision of a rational basis for pricing goods or services
3. Provision of a means for allocating and controlling expenditures
4. Provision of information on which operating decisions may be based and by means of which operating decisions may be evaluated

Although the basic objectives of cost accounting are simple, the exact determination of costs usually is not. As a result, some of the procedures used are arbitrary devices that make it possible to obtain reasonably accurate answers for most cases but that may contain a considerable percentage of error in other cases, particularly with respect to the actual cash flow involved.

2-A.3 Cost Accounting Example

This relatively simple example involves a job-order system in which costs are assigned to work by job number. Schematically, this process is illustrated in the following diagram:



Account		Transaction 1	2	3	4	5	6	7	Balances at End of Year
Assets	Cash	+$3,000	−$2,000	+$1,500	−$1,200		+$2,200	−$1,300	+$2,200
	Accounts receivable					+$3,000	−2,200		+800
	Inventory		+2,000		+2,100	−2,100			0
	Equipment				−500				+1,500
									$4,500
equals									
Liabilities	Accounts payable				+400			−300	+100
	Bank note			+1,500				−1,000	+500
plus *Owners' equity*	Equity	+3,000				+900			+3,900
									$4,500

Figure 2-A-1 Accounting Effects of Transactions: XYZ Firm

XYZ Firm Balance Sheet
as of December 31, 2010

Assets		Liabilities and Owners' Equity	
Cash	$2,200	Bank note	$500
Accounts receivable	800	Accounts payable	100
Equipment	1,500	Equity	3,900
Total	$4,500	Total	$4,500

XYZ Firm Income Statement
for Year Ending December 31, 2010

			Cash Flow
Operating revenues (Sales)		$3,000	$2,200
Operating costs (Inventory depleted)			
Labor	$1,200		−1,200
Material	400		−300
Depreciation	500		0
		$2,100	
Net income (Profits)		$900	$700

Figure 2-A-2 Balance Sheet and Income Statement Resulting from Transactions Shown in Figure 2-A-1

Costs are assigned to jobs in the following manner:

1. Raw materials attach to jobs via material requisitions.
2. Direct labor attaches to jobs via direct labor tickets.
3. Overhead cannot be attached to jobs directly but must have an allocation procedure that relates it to one of the resource factors, such as direct labor, which is already accumulated by the job.

Consider how an order for 100 tennis rackets accumulates costs at the Bowling Sporting Goods Company:

Job #161	100 tennis rackets
Labor rate	$7 per hour
Leather	50 yards at $2 per yard
Gut	300 yards at $0.50 per yard
Graphite	180 pounds at $3 per pound
Labor hours for the job	200 hours
Total annual factory overhead costs	$600,000
Total annual direct labor hours	200,000 hours

The three major costs are now attached to the job. Direct labor and material expenses are straightforward:

Job #161		
Direct labor	200 × $7 =	$1,400
Direct material	leather: 50 × $2 =	100
	gut: 300 × $0.5 =	150
	graphite: 180 × $3 =	540
Prime costs (direct labor + direct materials)		$2,190

Notice that this cost is not the total cost. We must somehow find a way to attach (allocate) factory costs that cannot be directly identified to the job but are nevertheless involved in producing the 100 rackets. Costs such as the power to run the graphite molding machine, the depreciation on this machine, the depreciation of the factory building, and the supervisor's salary constitute overhead for this company. These overhead costs are part of the cost structure of the 100 rackets but cannot be directly traced to the job. For instance, do we really know how much machine obsolescence is attributable to the 100 rackets? Probably not. Therefore, we must allocate these overhead costs to the 100 rackets by using the overhead rate determined as follows:

$$\text{Overhead rate} = \frac{\$600{,}000}{200{,}000} = \$3 \text{ per direct labor hour.}$$

This means that $600 ($3 × 200) of the total annual overhead cost of $600,000 would be allocated to Job #161. Thus, the total cost of Job #161 would be as follows:

Direct labor	$1,400
Direct materials	790
Factory overhead	600
	$2,790

The cost of manufacturing each racket is thus $27.90. If selling expenses and administrative expenses are allocated as 40% of the cost of goods sold, the total expense of a tennis racket becomes 1.4($27.90) = $39.06.

Appendix 2-A Problems

2-A-1.* Jill Smith opens an apartment-locator business near a college campus. She is the sole owner of the proprietorship, which she names Campus Apartment Locators. During the first month of operations, July 2010, she engages in the following transactions:

a. Smith invests $35,000 of personal funds to start the business.

b. She purchases on account office supplies costing $350.

c. Smith pays cash of $30,000 to acquire a lot next to the campus. She intends to use the land as a future building site for her business office.

d. Smith locates apartments for clients and receives cash of $1,900.

e. She pays $100 on the account payable she created in transaction (b).

f. She pays $2,000 of personal funds for a vacation.

g. She pays cash expenses for office rent, $400, and utilities, $100.

h. The business sells office supplies to another business for its cost of $150.

i. Smith withdraws cash of $1,200 for personal use.

Required

(a) Analyze the preceding transactions in terms of their effects on the accounting equation of Campus Apartment Locators. Use Figure 2-A-1 as a guide.

(b) Prepare the income statement and balance sheet of the business after recording the transactions. Use Figure 2-A-2 as a guide.

2-A-2.* Daniel Peavy owns and operates an architectural firm called Peavy Design. Table P2-A-2 summarizes the financial position of his business on April 30, 2010.

During May 2010, the following events occurred:

a. Peavy received $12,000 as a gift and deposited the cash in the business bank account.

b. He paid off the beginning balance of accounts payable.

c. He performed services for a client and received cash of $1,100.

d. He collected cash from a customer on account, $750.

e. Peavy purchased supplies on account, $720.

f. He consulted on the interior design of a major office building and billed the client for services rendered, $5,000.

g. He invested personal cash of $1,700 in the business.

h. He recorded the following business expenses for the month:

1. Paid office rent, $1,200.
2. Paid advertising, $660.

i. Peavy sold supplies to another interior designer for $80 cash, which was the cost of the supplies.

j. He withdrew cash of $4,000 for personal use.

Required

(a) Analyze the effects of the preceding transactions on the accounting equation of Peavy Design. Adapt the format of Figure 2-A-1.

(b) Prepare the income statement of Peavy Design for the month ended May 31, 2010. List expenses in decreasing order by amount.

(c) Prepare the balance sheet of Peavy Design at May 31, 2010.

Data for Problem 2-A-2

Assets				=	Liabilities	+	Owner's Equity
Cash	+ Accounts Receivable	+ Supplies	+ Land	=	Accounts Payable	+	Daniel Peavy, Capital
Bal. 1,720	3,240		24,100		5,400		23,660

2-A-3.* Lubbock Engineering Consultants is a firm of professional civil engineers. It mostly does surveying jobs for the heavy construction industry throughout Texas. The firm obtains its jobs by giving fixed-price quotations, so profitability depends on the ability to predict the time required for the various subtasks on the job. (This situation is similar to that in the auditing profession, where times are budgeted for such audit steps as reconciling cash and confirming accounts receivable.)

A client may be served by various professional staff, who hold positions in the hierarchy from partners to managers to senior engineers to assistants. In addition, there are secretaries and other employees.

Lubbock Engineering has the following budget for 2011:

Compensation of professional staff	$3,600,000
Other costs	1,449,000
Total budgeted costs	$5,049,000

* Adapted from C. T. Horngren, W. T. Harrison Jr., and L. S. Bamber, *Accounting*, 6th ed. (Upper Saddle River, NJ: Prentice Hall, 2005), p. 23 and p. 32. Reprinted by permission of the publisher.

* Adapted from C. T. Horngren, G. L. Sundem, and W. O. Stratton, *Introduction to Management Accounting*, 11th ed. (Upper Saddle River, NJ: Prentice-Hall, 1999), pp. 528–529. Reprinted by permission of the publisher.

3-35. Given the following information, how many units must be sold to achieve a profit of $25,000? [Note that the units sold must account for total production costs (direct and overhead) plus desired profit.] **(3.4, 3.5)**

Direct labor hours:	0.2 hour/unit
Direct labor costs:	$21.00/hour
Direct materials cost:	$4.00/unit
Overhead costs:	120% of direct labor
Packaging and shipping:	$1.20/unit
Selling price:	$20.00/unit

FE Practice Problems

3-36. Find the average time per unit required to produce the 30th unit, if the slope parameter of the learning rate is 90% and the first unit takes 450 hours.

(a) −3.30693E-11 (b) 305.5404 (c) 268
(d) 347.3211

3-37. A student is considering the purchase of two alternative cars. Car A initially costs $1,500 more than Car B, but uses 0.05 gallons per mile, versus 0.07 gallons per mile for Car B. Both cars will last for 10 years, and B's market value is $800 less than A's. Fuel costs $3.00 per gallon. If all else is equal, at how many miles driven per year does Car A become preferable to Car B?

(a) 1,167 (b) 1,723 (c) 1,892
(d) 2,243

3-38. An automatic process controller will eliminate the current manual control operation. Annual cost of the current method is $4,000. If the controller has a service life of 13 years and an expected market value of 11% of the first cost, what is the maximum economical price for the controller? Ignore interest.

(a) $28,869 (b) $58,426 (c) $26,358
(d) $25,694 (e) $53,344

3-39. A foreman supervises A, B, and eight other employees. The foreman states that he spends twice as much time supervising A and half as much time supervising B, compared with the average time spent supervising his other subordinates. All employees have the same production rate. On the basis of equal cost per unit production, what monthly salary is justified for B if the foreman gets $3,800 per month and A gets $3,000 per month?

(a) $3,543 (b) $3,800 (c) $3,000
(d) $2,457 (e) $3,400

3-40. An automatic process controller will eliminate the current manual control operations. The annual cost of the current method is $5,000. If the controller has a service life of 12 years, and an expected market value of 10% of the first cost, what is the maximum economical price for the controller? Ignore interest.

(a) $28,869 (b) $58,426 (c) $26,358
(d) $25,694 (e) $66,666.67

3-41. A small textile plant was constructed in 2000. The major equipment, costs, and factors are shown below.

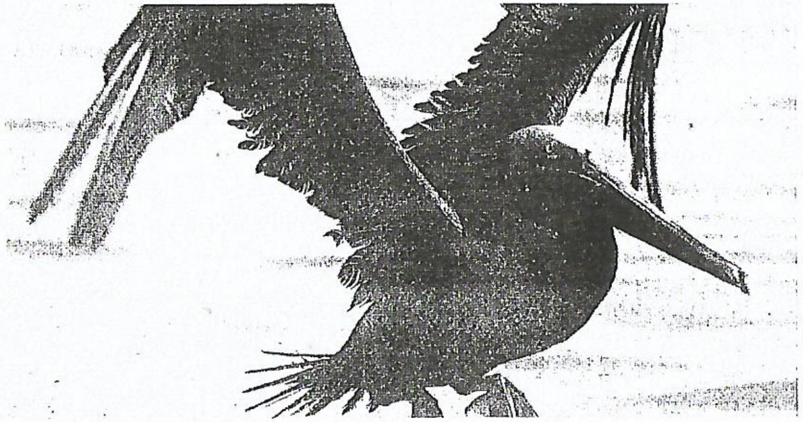
Estimate the cost to build a new plant in 2010 if the index for this type of equipment has increased at an average rate of 12% per year for the past 10 years. Select the closest answer. **(3.4)**

(a) $4,618,000 (b) $10,623,000
(c) $14,342,000 (d) $14,891,000

Equipment	Reference Size	Reference Cost	Cost-Capacity Factor	New Design Size
Finishing machine	150 yd/min	$900,000	0.92	200 yd/min
Jet dyer	200 yd/min	$1,125,000	0.87	450 yd/min
Steam dyer	100 yd/min	$750,000	0.79	175 yd/min

CHAPTER 4

The Time Value of Money



The primary focus of Chapter 4 is to explain time value of money calculations and to illustrate economic equivalence.

Crisis in the Gulf

How vividly do you remember the biggest man-made environmental catastrophe in American history—millions of gallons of oil flowing unchecked into the Gulf of Mexico from an undersea well? In response to this tragedy, British Petroleum (BP) will make payments into a fund to pay for some of the damages to the Gulf Coast resulting from their massive oil spill in April and following months of 2010. BP will pay $3 billion at the end of the third quarter of 2010 and another $2 billion in the fourth quarter of 2010. BP will then make payments of $1.25 billion each quarter thereafter until a total of $20 billion has been paid into the fund. If the opportunity cost of capital (interest rate) is 3% per quarter, what is the equivalent value of this payment stream at the beginning of the third quarter of 2010? This is one type of problem you can answer after studying Chapter 4. We will return to this problem in Example 4-18.

The Importance of Interest in Your Daily Life

In 2005, total debt (credit cards, auto loans, home mortgages, etc.) amounted to *more* than 100% of total disposable income for the average U.S. household. If your total disposable income is \$50,000, how much interest can you expect to pay when the average interest rate on your debt is 12% per year?

Answer: You can expect to pay \$50,000 (0.12) = \$6,000 per year!

4.3 Compound Interest

Whenever the interest charge for any interest *period* (a year, for example) is based on the remaining principal amount plus any accumulated interest charges up to the *beginning* of that period, the interest is said to be *compound*. The effect of compounding of interest can be seen in the following table for \$1,000 loaned for three periods at an interest rate of 10% compounded each period:

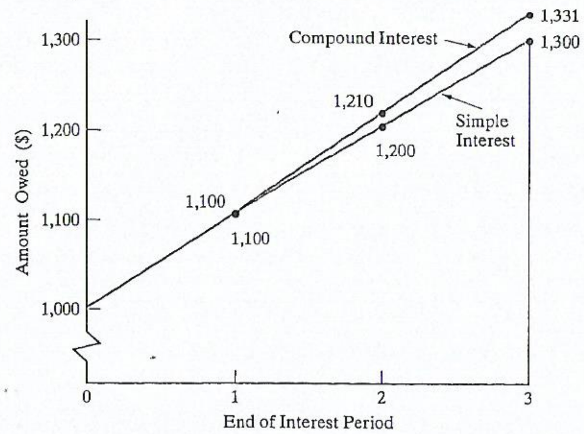
Period	(1) Amount Owed at Beginning of Period	(2) = (1) × 10% Interest Amount for Period	(3) = (1) + (2) Amount Owed at End of Period
1	\$1,000	\$100	\$1,100
2	\$1,100	\$110	\$1,210
3	\$1,210	\$121	\$1,331

As you can see, a total of \$1,331 would be due for repayment at the end of the third period. If the length of a period is one year, the \$1,331 at the end of three periods (years) can be compared with the \$1,300 given earlier for the same problem with simple interest. A graphical comparison of simple interest and compound interest is given in Figure 4-1. The difference is due to the effect of *compounding*, which is essentially the calculation of interest on previously earned interest. This difference would be much greater for larger amounts of money, higher interest rates, or greater numbers of interest periods. Thus, simple interest does consider the time value of money but does not involve compounding of interest. Compound interest is much more common in practice than simple interest and is used throughout the remainder of this book.

4.4 The Concept of Equivalence

Alternatives should be compared when they produce similar results, serve the same purpose, or accomplish the same function. This is not always possible in some types of economy studies (as we shall see later), but now our attention is directed at answering the question: How can alternatives for providing the same service or accomplishing the same function be compared when interest is involved over extended periods of time? Thus, we should consider the comparison of alternative

Figure 4-1 Illustration of Simple versus Compound Interest



options, or proposals, by reducing them to an *equivalent basis* that is dependent on (1) the interest rate, (2) the amounts of money involved, and (3) the timing of the monetary receipts or expenses.

To better understand the mechanics of interest and to explain the concept of equivalence, suppose you have a \$17,000 balance on your credit card. "This has got to stop!" you say to yourself. So you decide to repay the \$17,000 debt in four months. An unpaid credit card balance at the beginning of a month will be charged interest at the rate of 1% by your credit card company. For this situation, we have selected three plans to repay the \$17,000 principal plus interest owed.* These three plans are illustrated in Table 4-1, and we will demonstrate that they are equivalent (i.e., the same) when the interest rate is 1% per month on the unpaid balance of principal.

Plan 1 indicates that none of the principal is repaid until the end of the fourth month. The monthly payment of interest is \$170, and all of the principal is also repaid at the end of month four. Because interest does not accumulate in Plan 1, compounding of interest is not present in this situation. In Table 4-1, there are 68,000 dollar-months of borrowing (\$17,000 × 4 months) and \$680 total interest. Therefore, the monthly interest rate is (\$680 ÷ 68,000 dollar-months) × 100% = 1%.

Plan 2 stipulates that we repay \$4,357.10 per month. Later we will show how this number is determined (Section 4.9). For our purposes here, you should observe that interest is being compounded and that the \$17,000 principal is completely repaid over the four months. From Table 4-1, you can see that the monthly interest rate is (\$427.10 ÷ 42,709.5 dollar-months of borrowing) × 100% = 1%. There are fewer dollar-months of borrowing in Plan 2 (as compared with Plan 1) because principal is being repaid every month and the total amount of interest paid (\$427.10) is less.

* These repayment plans are for demonstration purposes *only*. It is very unlikely that a credit card company would agree to either Plan 1 or Plan 3 without additional charges and/or damaging your credit history.

Three Plans for Repayment of $17,000 in Four Months with Interest at 1% per Month

(1) Month	(2) Amount Owed at Beginning of Month	(3) = 1% × (2) Interest Accrued for Month	(4) = (2) + (3) Total Money Owed at End of Month	(5) Principal Payment	(6) = (3) + (5) Total End-of-Month Payment (Cash Flow)
Plan 1: Pay interest due at end of each month and principal at end of fourth month.					
1	$17,000	$170	$17,170	$0	$170
2	17,000	170	17,170	0	170
3	17,000	170	17,170	0	170
4	17,000	170	17,170	17,000	17,170
	68,000 $-mo.	$680 (total interest)			
Plan 2: Pay off the debt in four equal end-of-month installments (principal and interest).					
1	$17,000	$170	$17,170	$4,187.10	$4,357.10
2	12,812.90	128.13	12,941.03	4,228.97	4,357.10
3	8,583.93	85.84	8,669.77	4,271.26	4,357.10
4	4,312.67	43.13	4,355.80	4,313.97	4,357.10
	42,709.5 $-mo.	$427.10 (total interest)	Difference = $1.30 due to roundoff		
Plan 3: Pay principal and interest in one payment at end of fourth month.*					
1	$17,000	$170	$17,170	$0	$0
2	17,170	171.70	17,341.70	0	0
3	17,341.70	173.42	17,515.12	0	0
4	17,515.12	175.15	17,690.27	17,000	17,690.27
	69,026.8 $-mo.	$690.27 (total interest)			

* Here, column 6 ≠ column 3 + column 5.

Finally, Plan 3 shows that no interest and no principal are repaid in the first three months. Then at the end of month four, a single lump-sum amount of $17,690.27 is repaid. This includes the original principal and the accumulated (compounded) interest of $690.27. The dollar-months of borrowing are very large for Plan 3 (69,026.8) because none of the principal and accumulated interest is repaid until the end of the fourth month. Again, the ratio of total interest paid to dollar-months is 0.01.

This brings us back to the concept of economic equivalence. If the interest rate remains at 1% per month, you should be indifferent as to which plan you use to repay the $17,000 to your credit card company. This assumes that you are charged 1% of the outstanding principal balance (which includes any unpaid interest) each month for the next four months. If interest rates in the economy go up and increase your credit card rate to say, $1\frac{1}{4}$% per month, the plans are no longer equivalent. What varies among the three plans is the rate at which principal is repaid and how interest is repaid.

4.5 Notation and Cash-Flow Diagrams and Tables

The following notation is utilized in formulas for compound interest calculations:

i = effective interest rate per interest period;
N = number of compounding (interest) periods;
P = present sum of money; the *equivalent* value of one or more cash flows at a reference point in time called the present;
F = future sum of money; the *equivalent* value of one or more cash flows at a reference point in time called the future;
A = end-of-period cash flows (or *equivalent* end-of-period values) in a uniform series continuing for a specified number of periods, starting at the end of the first period and continuing through the last period.

The use of cash-flow (time) diagrams or tables is strongly recommended for situations in which the analyst needs to clarify or visualize what is involved when flows of money occur at various times. In addition, viewpoint (remember Principle 3?) is an essential feature of cash-flow diagrams.

The difference between total cash inflows (receipts) and cash outflows (expenditures) for a specified period of time (e.g., one year) is the net cash flow for the period. As discussed in Chapters 2 and 3, cash flows are important in engineering economy because they form the basis for evaluating alternatives. Indeed, the usefulness of a cash-flow diagram for economic analysis problems is analogous to that of the free-body diagram for mechanics problems.

Figure 4-2 shows a cash-flow diagram for Plan 3 of Table 4-1, and Figure 4-3 depicts the net cash flows of Plan 2. These two figures also illustrate the definition of the preceding symbols and their placement on a cash-flow diagram. Notice that all cash flows have been placed at the end of the month to correspond with the convention used in Table 4-1. In addition, a viewpoint has been specified.

The cash-flow diagram employs several conventions:

1. The horizontal line is a *time scale*, with progression of time moving from left to right. The period (e.g., year, quarter, month) labels can be applied to intervals of time rather than to points on the time scale. Note, for example, that the end of Period 2 is coincident with the beginning of Period 3. When the end-of-period cash-flow convention is used, period numbers are placed at the end of each time interval, as illustrated in Figures 4-2 and 4-3.

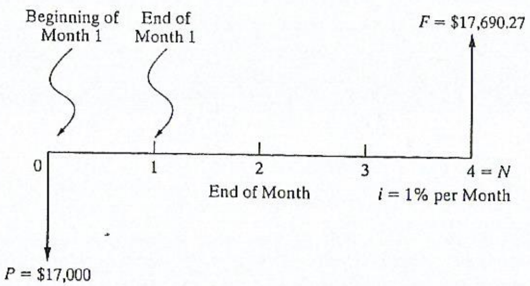
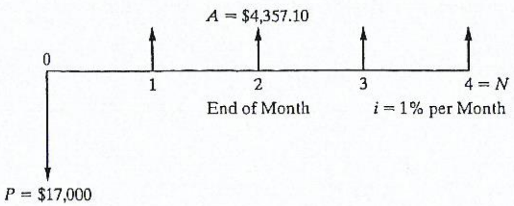


Figure 4-2 Cash-Flow Diagram for Plan 3 of Table 4-1 (Credit Card Company's Viewpoint)

Figure 4-3 Cash-Flow Diagram for Plan 2 of Table 4-1 (Lender's Viewpoint)



2. The arrows signify cash flows and are placed at the end of the period. If a distinction needs to be made, downward arrows represent expenses (negative cash flows or cash outflows) and upward arrows represent receipts (positive cash flows or cash inflows).
3. The cash-flow diagram is dependent on the point of view. For example, the situations shown in Figures 4-2 and 4-3 were based on cash flow as seen by the lender (the credit card company). If the directions of all arrows had been reversed, the problem would have been diagrammed from the borrower's viewpoint.

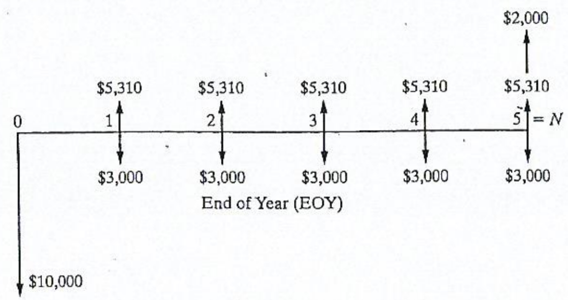
EXAMPLE 4-1

Cash-Flow Diagramming

Before evaluating the economic merits of a proposed investment, the XYZ Corporation insists that its engineers develop a cash-flow diagram of the proposal. An investment of $10,000 can be made that will produce uniform annual revenue of $5,310 for five years and then have a market (recovery) value of $2,000 at the end of year (EOY) five. Annual expenses will be $3,000 at the end of each year for operating and maintaining the project. Draw a cash-flow diagram for the five-year life of the project. Use the corporation's viewpoint.

Solution

As shown in the figure below, the initial investment of $10,000 and annual expenses of $3,000 are cash outflows, while annual revenues and the market value are cash inflows.



Notice that the beginning of a given year is the end of the preceding year. For example, the beginning of year two is the end of year one.

Example 4-2 presents a situation in which cash flows are represented in tabular form to facilitate the analysis of plans and designs.

EXAMPLE 4-2

Developing a Net Cash-Flow Table

In a company's renovation of a small office building, two feasible alternatives for upgrading the heating, ventilation, and air conditioning (HVAC) system have been identified. Either Alternative *A* *or* Alternative *B* must be implemented. The costs are as follows:

Alternative *A*	*Rebuild (overhaul) the existing HVAC system*	
	• Equipment, labor, and materials to rebuild	$18,000
	• Annual cost of electricity	32,000
	• Annual maintenance expenses	2,400
Alternative *B*	*Install a new HVAC system that utilizes existing ductwork*	
	• Equipment, labor, and materials to install	$60,000
	• Annual cost of electricity	9,000
	• Annual maintenance expenses	16,000
	• Replacement of a major component four years hence	9,400

At the end of eight years, the estimated market value for Alternative *A* is $2,000 and for Alternative *B* it is $8,000. Assume that both alternatives will provide comparable service (comfort) over an eight-year period, and assume that the major component replaced in Alternative *B* will have no market value at EOY eight. (1) Use a cash-flow table and end-of-year convention to tabulate the net cash flows for both alternatives. (2) Determine the annual net cash-flow difference between the alternatives ($B - A$).

Solution

The cash-flow table (company's viewpoint) for this example was developed by using a spreadsheet and is shown in Figure 4-4. On the basis of these results, several points can be made: (1) Doing nothing is not an option—either *A* or *B* must be selected; (2) even though positive and negative cash flows are included in the table, on balance we are investigating two "cost-only" alternatives; (3) a decision between the two alternatives can be made just as easily on the *difference* in cash flows (i.e., on the avoidable difference) as it can on the stand-alone net cash flows for Alternatives *A* and *B*; (4) Alternative *B* has cash flows identical to those of Alternative *A*, *except for* the differences shown in the table, so if the avoidable difference can "pay its own way," Alternative *B* is the recommended choice; (5) cash-flow changes caused by inflation or other suspected influences could have easily been inserted into the table and included in the analysis; and

= – 25000 – 9400 | = C3 – B3 | = SUM(D$3:D$3)

	A	B	C	D	E
1		Alternative A	Alternative B	Difference	Cumulative
2	End of Year	Net Cash Flow	Net Cash Flow	(B-A)	Difference
3	0 (now)	$ (18,000)	$ (60,000)	$ (42,000)	$ (42,000)
4	1	$ (34,400)	$ (25,000)	$ 9,400	$ (32,600)
5	2	$ (34,400)	$ (25,000)	$ 9,400	$ (23,200)
6	3	$ (34,400)	$ (25,000)	$ 9,400	$ (13,800)
7	4	$ (34,400)	$ (34,400)	$ -	$ (13,800)
8	5	$ (34,400)	$ (25,000)	$ 9,400	$ (4,400)
9	6	$ (34,400)	$ (25,000)	$ 9,400	$ 5,000
10	7	$ (34,400)	$ (25,000)	$ 9,400	$ 14,400
11	8	$ (32,400)	$ (17,000)	$ 15,400	$ 29,800
12	Total	$ (291,200)	$ (261,400)		

= – 34400 + 2000

= – 25000 + 8000

= SUM(B3:B11)

Figure 4-4 Cash-Flow Table, Example 4-2

(6) it takes six *years* for the extra $42,000 investment in Alternative *B* to generate sufficient cumulative savings in annual expenses to justify the higher investment. (This ignores the time value of money.) So, which alternative is better? We'll be able to answer this question later when we consider the time value of money in order to recommend choices between alternatives.

Comment

Cash-flow tables are invaluable when using a spreadsheet to model engineering economy problems.

It should be apparent that a cash-flow table clarifies the timing of cash flows, the assumptions that are being made, and the data that are available. A cash-flow table is often useful when the complexity of a situation makes it difficult to show all cash-flow amounts on a diagram.

The remainder of Chapter 4 deals with the development and illustration of equivalence (time value of money) principles for assessing the economic attractiveness of investments, such as those proposed in Examples 4-1 and 4-2.

Viewpoint: In most examples presented in this chapter, the company's (investor's) viewpoint will be taken.

4.6 Relating Present and Future Equivalent Values of Single Cash Flows

Figure 4-5 shows a cash-flow diagram involving a present single sum, P, and a future single sum, F, separated by N periods with interest at i% per period. Throughout this chapter, a *dashed arrow*, such as that shown in Figure 4-5, indicates the quantity to be determined.

4.6.1 Finding *F* when Given *P*

If an amount of P dollars is invested at a point in time and i% is the interest (profit or growth) rate per period, the amount will grow to a future amount of $P + Pi = P(1 + i)$ by the end of one period; by the end of two periods, the amount will grow to $P(1 + i)(1 + i) = P(1 + i)^2$; by the end of three periods, the amount will grow to $P(1 + i)^2(1 + i) = P(1 + i)^3$; and by the end of N periods the amount will grow to

$$F = P(1 + i)^N. \tag{4-2}$$

EXAMPLE 4-3 **Future Equivalent of a Present Sum**

Suppose that you borrow $8,000 now, promising to repay the loan principal plus accumulated interest in four years at $i = 10\%$ per year. How much would you repay at the end of four years?

Solution

Year	Amount Owed at Start of Year	Interest Owed for Each Year	Amount Owed at End of Year	Total End-of-Year Payment
1	$P = \$\ 8{,}000$	$iP = \$\ 800$	$P(1+i) = \$\ 8{,}800$	0
2	$P(1+i) = \$\ 8{,}800$	$iP(1+i) = \$\ 880$	$P(1+i)^2 = \$\ 9{,}680$	0
3	$P(1+i)^2 = \$\ 9{,}680$	$iP(1+i)^2 = \$\ 968$	$P(1+i)^3 = \$10{,}648$	0
4	$P(1+i)^3 = \$10{,}648$	$iP(1+i)^3 = \$1{,}065$	$P(1+i)^4 = \$11{,}713$	$F = \$11{,}713$

In general, we see that $F = P(1+i)^N$, and the total amount to be repaid is $11,713.

The quantity $(1 + i)^N$ in Equation (4-2) is commonly called the *single payment compound amount factor*. Numerical values for this factor are given in the second column from the left in the tables of Appendix C for a wide range of values of i and N. In this book, we shall use the functional symbol $(F/P, i\%, N)$ for $(1 + i)^N$. Hence, Equation (4-2) can be expressed as

$$F = P(F/P, i\%, N). \tag{4-3}$$

where the factor in parentheses is read "find F given P at i% interest per period for N interest periods." Note that the sequence of F and P in F/P is the same as in

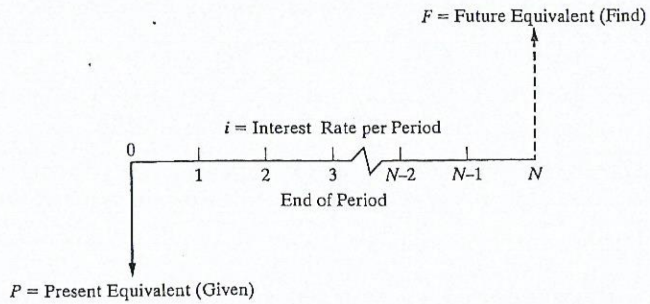


Figure 4-5 General Cash-Flow Diagram Relating Present Equivalent and Future Equivalent of Single Payments

the initial part of Equation (4-3), where the unknown quantity, F, is placed on the left-hand side of the equation. This sequencing of letters is true of all functional symbols used in this book and makes them easy to remember.

Let's look at Example 4-3 again. Using Equation (4-3) and Appendix C, we have

$$F = \$8{,}000(F/P, 10\%, 4)$$
$$= \$8{,}000(1.4641)$$
$$= \$11{,}713.$$

This, of course, is the same result obtained in Example 4-3 since $(F/P,\ 10\%,\ 4) = (1 + 0.10)^4 = 1.4641$.

Another example of finding F when given P, together with a cash-flow diagram and solution, appears in Table 4-2. Note in Table 4-2 that, for each of the six common discrete compound interest circumstances covered, two problem statements are given—(1) *in borrowing–lending terminology* and (2) *in equivalence terminology*—but they both represent the same cash-flow situation. Indeed, there are generally many ways in which a given cash-flow situation can be expressed.

In general, a good way to interpret a relationship such as Equation (4-3) is that the calculated amount, F, at the point in time at which it occurs, *is equivalent to* (i.e., can be traded for) the known value, P, at the point in time at which it occurs, for the given interest or profit rate, i.

4.6.2 Finding *P* when Given *F*

From Equation (4-2), $F = P(1 + i)^N$. Solving this for P gives the relationship

$$P = F\left(\frac{1}{1+i}\right)^N = F(1+i)^{-N}. \qquad (4\text{-}4)$$

The quantity $(1 + i)^{-N}$ is called the *single payment present worth factor*. Numerical values for this factor are given in the third column of the tables in Appendix C for

a wide range of values of i and N. We shall use the functional symbol $(P/F,\ i\%,\ N)$ for this factor. Hence,

$$P = F(P/F,\ i\%,\ N). \qquad (4\text{-}5)$$

EXAMPLE 4-4 **Present Equivalent of a Future Amount of Money**

An investor (owner) has an option to purchase a tract of land that will be worth \$10,000 in six years. If the value of the land increases at 8% each year, how much should the investor be willing to pay now for this property?

Solution

The purchase price can be determined from Equation (4-5) and Table C-11 in Appendix C as follows:

$$P = \$10{,}000(P/F, 8\%, 6)$$
$$P = \$10{,}000(0.6302)$$
$$= \$6{,}302.$$

Another example of this type of problem, together with a cash-flow diagram and solution, is given in Table 4-2.

Based on Equations (4-2) and (4-4), the following three simple rules apply when performing arithmetic calculations with cash flows:

Rule A. Cash flows cannot be added or subtracted unless they occur at the same point in time.

Rule B. To move a cash flow forward in time by one time unit, multiply the magnitude of the cash flow by $(1 + i)$, where i is the interest rate that reflects the time value of money.

Rule C. To move a cash flow backward in time by one time unit, divide the magnitude of the cash flow by $(1 + i)$.

4.6.3 Finding the Interest Rate Given *P*, *F*, and *N*

There are situations in which we know two sums of money (P and F) and how much time separates them (N), but we don't know the interest rate (i) that makes them equivalent. For example, if we want to turn \$500 into \$1,000 over a period of 10 years, at what interest rate would we have to invest it? We can easily solve Equation (4-2) to obtain an expression for i.

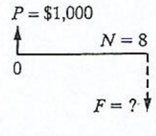
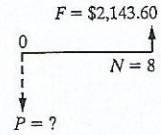
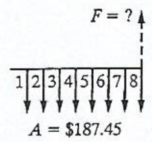
$$i = \sqrt[N]{F/P} - 1 \qquad (4\text{-}6)$$

So, for our simple example, $i = \sqrt[10]{\$1{,}000/\$500} - 1 = 0.0718$ or 7.18% per year.

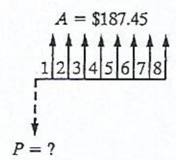
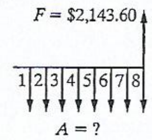
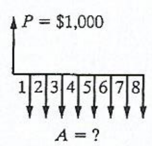
Inflation is another example of when it may be necessary to solve for an interest rate. Suppose you are interested in determining the annual rate of increase in the

Discrete Cash-Flow Examples Illustrating Equivalence

Example Problems (All Using an Interest Rate of $i = 10\%$ per Year—See Table C-13 of Appendix C)

To Find:	Given:	(a) In Borrowing–Lending Terminology:	(b) In Equivalence Terminology:	Cash-Flow Diagram[a]	Solution
For single cash flows:					
F	P	A firm borrows $1,000 for eight years. How much must it repay in a lump sum at the end of the eighth year?	What is the future equivalent at the end of eight years of $1,000 at the beginning of those eight years?		$F = P(F/P, 10\%, 8)$ $= \$1{,}000(2.1436)$ $= \$2{,}143.60$
P	F	A firm wishes to have $2,143.60 eight years from now. What amount should be deposited now to provide for it?	What is the present equivalent of $2,143.60 received eight years from now?		$P = F(P/F, 10\%, 8)$ $= \$2{,}143.60(0.4665)$ $= \$1{,}000.00$
For uniform series:					
F	A	If eight annual deposits of $187.45 each are placed in an account, how much money has accumulated immediately after the last deposit?	What amount at the end of the eighth year is equivalent to eight EOY payments of $187.45 each?		$F = A(F/A, 10\%, 8)$ $= \$187.45(11.4359)$ $= \$2{,}143.60$

(Continued)

To Find:	Given:	(a) In Borrowing–Lending Terminology:	(b) In Equivalence Terminology:	Cash-Flow Diagram[a]	Solution
P	A	How much should be deposited in a fund now to provide for eight EOY withdrawals of $187.45 each?	What is the present equivalent of eight EOY payments of $187.45 each?		$P = A(P/A, 10\%, 8)$ $= \$187.45(5.3349)$ $= \$1{,}000.00$
A	F	What uniform annual amount should be deposited each year in order to accumulate $2,143.60 at the time of the eighth annual deposit?	What uniform payment at the end of eight successive years is equivalent to $2,143.60 at the end of the eighth year?		$A = F(A/F, 10\%, 8)$ $= \$2{,}143.60(0.0874)$ $= \$187.45$
A	P	What is the size of eight equal annual payments to repay a loan of $1,000? The first payment is due one year after receiving the loan.	What uniform payment at the end of eight successive years is equivalent to $1,000 at the beginning of the first year?		$A = P(A/P, 10\%, 8)$ $= \$1{,}000(0.18745)$ $= \$187.45$

[a] The cash-flow diagram represents the example as stated in borrowing-lending terminology.

price of gasoline. Given the average prices in different years, you can use the relationship between P and F to solve for the inflation rate.

EXAMPLE 4-5

The Inflating Price of Gasoline

The average price of gasoline in 2005 was \$2.31 per gallon. In 1993, the average price was \$1.07.* What was the average annual rate of increase in the price of gasoline over this 12-year period?

Solution

With respect to the year 1993, the year 2005 is in the future. Thus, $P = \$1.07$, $F = \$2.31$, and $N = 12$. Using Equation (4-6), we find $i = \sqrt[12]{2.31/1.07} - 1 = 0.0662$ or 6.62% per year.

* This data was obtained from the Energy Information Administration of the Department of Energy. Historical prices of gasoline and other energy sources can be found at www.eia.doe.gov.

4.6.4 Finding N when Given P, F, and i

Sometimes we are interested in finding the amount of time needed for a present sum to grow into a future sum at a specified interest rate. For example, how long would it take for \$500 invested today at 15% interest per year to be worth \$1,000? We can use the equivalence relationship given in Equation (4-2) to obtain an expression for N.

$$F = P(1+i)^N$$

$$(1+i)^N = (F/P)$$

Using logarithms,

$$N \log(1+i) = \log(F/P)$$

and

$$N = \frac{\log(F/P)}{\log(1+i)}. \qquad (4\text{-}7)$$

For our simple example, $N = \log(\$1{,}000/\$500)/\log(1.15) = 4.96 \cong 5$ years.

EXAMPLE 4-6

When Will Gasoline Cost \$5.00 per Gallon?

In Example 4-5, the average price of gasoline was given as \$2.31 in 2005. We computed the average annual rate of increase in the price of gasoline to be 6.62%. If we assume that the price of gasoline will continue to inflate at this rate, how long will it be before we are paying \$5.00 per gallon?

Solution

We have $P = \$2.31$, $F = \$5.00$, and $i = 6.62\%$ per year. Using Equation (4-7), we find

$$N = \frac{\log(\$5.00/\$2.31)}{\log(1+0.0662)} = \frac{\log(2.1645)}{\log(1.0662)} = 12.05 \text{ years.}$$

So, if gasoline prices continue to increase at the same rate, we can expect to be paying \$5.00 per gallon in 2017.

4.7 Relating a Uniform Series (Annuity) to Its Present and Future Equivalent Values

Figure 4-6 shows a general cash-flow diagram involving a series of uniform (equal) receipts, each of amount A, occurring at the end of each period for N periods with interest at i% per period. Such a uniform series is often called an *annuity*. It should be noted that the formulas and tables to be presented are derived such that A occurs at the end of each period, and thus,

1. P (present equivalent value) occurs one interest period before the first A (uniform amount),
2. F (future equivalent value) occurs at the same time as the last A, and N periods after P, and
3. A (annual equivalent value) occurs at the end of periods 1 through N, inclusive.

The timing relationship for P, A, and F can be observed in Figure 4-6. Four formulas relating A to F and P will be developed.

4.7.1 Finding F when Given A

If a cash flow in the amount of A dollars occurs at the end of each period for N periods and i% is the interest (profit or growth) rate per period, the future equivalent value, F, at the end of the Nth period is obtained by summing the future equivalents

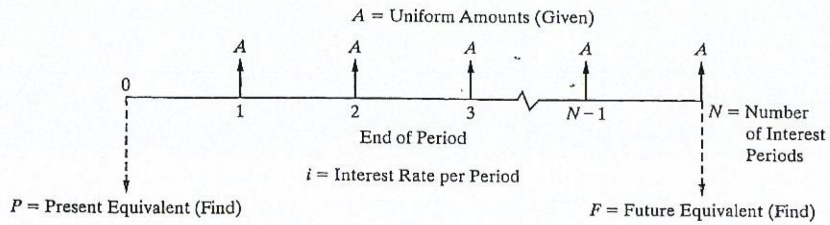


Figure 4-6 General Cash-Flow Diagram Relating Uniform Series (Ordinary Annuity) to Its Present Equivalent and Future Equivalent Values

of each of the cash flows. Thus,

$$\begin{aligned} F &= A(F/P, i\%, N-1) + A(F/P, i\%, N-2) + A(F/P, i\%, N-3) + \cdots \\ &\quad + A(F/P, i\%, 1) + A(F/P, i\%, 0) \\ &= A[(1+i)^{N-1} + (1+i)^{N-2} + (1+i)^{N-3} + \cdots + (1+i)^1 + (1+i)^0]. \end{aligned}$$

The bracketed terms comprise a geometric sequence having a common ratio of $(1+i)^{-1}$. Recall that the sum of the first N terms of a geometric sequence is

$$S_N = \frac{a_1 - b a_N}{1-b} \quad (b \neq 1),$$

where a_1 is the first term in the sequence, a_N is the last term, and b is the common ratio. If we let $b = (1+i)^{-1}$, $a_1 = (1+i)^{N-1}$, and $a_N = (1+i)^0$, then

$$F = A\left[\frac{(1+i)^{N-1} - \dfrac{1}{(1+i)}}{1 - \dfrac{1}{(1+i)}}\right],$$

which reduces to

$$F = A\left[\frac{(1+i)^N - 1}{i}\right]. \tag{4-8}$$

The quantity $\{[(1+i)^N - 1]/i\}$ is called the *uniform series compound amount factor.* It is the starting point for developing the remaining three uniform series interest factors.

Numerical values for the uniform series compound amount factor are given in the fourth column of the tables in Appendix C for a wide range of values of i and N. We shall use the functional symbol $(F/A, i\%, N)$ for this factor. Hence, Equation (4-8) can be expressed as

$$F = A(F/A, i\%, N). \tag{4-9}$$

Examples of this type of "wealth accumulation" problem based on the $(F/A, i\%, N)$ factor are provided here and in Table 4-2.

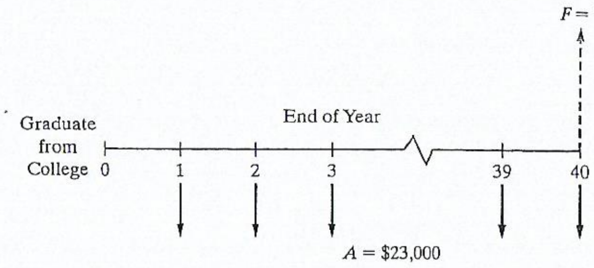
EXAMPLE 4-7

Future Value of a College Degree

A recent government study reported that a college degree is worth an extra \$23,000 per year in income (A) compared to what a high-school graduate makes. If the interest rate (i) is 6% per year and you work for 40 years (N), what is the future compound amount (F) of this extra income?

Solution

The viewpoint we will use to solve this problem is that of "lending" the \$23,000 of extra annual income to a savings account (or some other investment vehicle). The future equivalent is the amount that can be withdrawn after the 40th deposit is made.



Notice that the future equivalent occurs at the *same time* as the last deposit of \$23,000.

$$\begin{aligned} F &= \$23{,}000(F/A, 6\%, 40) \\ &= \$23{,}000(154.762) \\ &= \$3{,}559{,}526 \end{aligned}$$

The bottom line is "Get your college degree!"

EXAMPLE 4-8

Become a Millionaire by Saving \$1.00 a Day!

To illustrate further the amazing effects of compound interest, we consider the credibility of this statement: "If you are 20 years of age and save \$1.00 each day for the rest of your life, you can become a millionaire." Let's assume that you live to age 80 and that the annual interest rate is 10% ($i = 10\%$). Under these specific conditions, we compute the future compound amount (F) to be

$$\begin{aligned} F &= \$365/\text{year } (F/A, 10\%, 60 \text{ years}) \\ &= \$365\ (3{,}034.81) \\ &= \$1{,}107{,}706. \end{aligned}$$

Thus, the statement is true for the assumptions given! The moral is to *start saving early* and let the "magic" of compounding work on your behalf!

A few words to the wise: Saving money early and preserving resources through frugality (avoiding waste) are extremely important ingredients of *wealth creation* in general. Often, being frugal means postponing the satisfaction of immediate material wants for the creation of a better tomorrow. In this regard, be very *cautious* about spending tomorrow's cash today by undisciplined borrowing (e.g., with credit cards). The $(F/A, i\%, N)$ factor also demonstrates how *fast* your debt can accumulate!

4.7.2 Finding P when Given A

From Equation (4-2), $F = P(1 + i)^N$. Substituting for F in Equation (4-8) we determine that

$$P(1+i)^N = A\left[\frac{(1+i)^N - 1}{i}\right].$$

Dividing both sides by $(1 + i)^N$, we get

$$P = A\left[\frac{(1+i)^N - 1}{i(1+i)^N}\right]. \tag{4-10}$$

Thus, Equation (4-10) is the relation for finding the present equivalent value (as of the beginning of the first period) of a uniform series of end-of-period cash flows of amount A for N periods. The quantity in brackets is called the *uniform series present worth factor*. Numerical values for this factor are given in the fifth column of the tables in Appendix C for a wide range of values of i and N. We shall use the functional symbol $(P/A, i\%, N)$ for this factor. Hence,

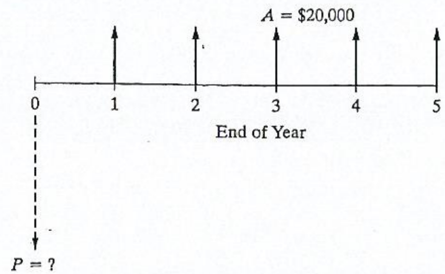
$$P = A(P/A, i\%, N). \tag{4-11}$$

EXAMPLE 4-9 Present Equivalent of an Annuity (Uniform Series)

If a certain machine undergoes a major overhaul now, its output can be increased by 20%—which translates into additional cash flow of $20,000 at the end of each year for five years. If $i = 15\%$ per year, how much can we afford to invest to overhaul this machine?

Solution

In the cash-flow diagram below, notice that the present equivalent, P, occurs one time period (year) *before* the first cash flow of $20,000.



The increase in cash flow is $20,000 per year, and it continues for five years at 15% annual interest. The upper limit on what we can afford to spend now is

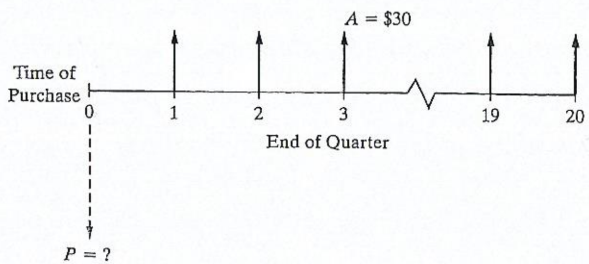
$$\begin{aligned} P &= \$20{,}000(P/A, 15\%, 5) \\ &= \$20{,}000(3.3522) \\ &= \$67{,}044. \end{aligned}$$

EXAMPLE 4-10 How Much Is a Lifetime Oil Change Offer Worth?

"Make your best deal with us on a new automobile and we'll change your oil for free for as long as you own the car!" If you purchase a car from this dealership, you expect to have four free oil changes per year during the five years you keep the car. Each oil change would normally cost you $30. If you save your money in a mutual fund earning 2% per quarter, how much are the oil changes worth to you at the time you buy the car?

Solution

In this example, we need to find the present equivalent of the cost of future oil changes. The cash-flow diagram is shown below. Notice that P occurs one time period (a quarter of a year, in this example) before the first oil change cash flow (A).



The interest rate is 2% per quarter, and a total of (4 oil changes/year × 5 years) = 20 oil changes (cash flows) are anticipated.

$$\begin{aligned} P &= \$30(P/A, 2\%, 20) \\ &= \$30(16.3514) \\ &= \$490.54 \end{aligned}$$

Now you are in a position to determine how great of a deal you are being offered. If the best price of another dealership is more than $490.54 cheaper than what you are being offered at this dealership, maybe this deal isn't so great.

4.7.3 Finding A when Given F

Taking Equation (4-8) and solving for A, we find that

$$A = F\left[\frac{i}{(1+i)^N - 1}\right]. \tag{4-12}$$

Thus, Equation (4-12) is the relation for finding the amount, A, of a uniform series of cash flows occurring at the end of N interest periods that would be equivalent to (have the same value as) its future value occurring at the end of the last period. The quantity in brackets is called the *sinking fund factor*. Numerical values for this factor are given in the sixth column of the tables in Appendix C for a wide range of values of i and N. We shall use the functional symbol $(A/F, i\%, N)$ for this factor. Hence,

$$A = F(A/F, i\%, N). \tag{4-13}$$

Another example of this type of problem, together with a cash-flow diagram and solution, is given in Table 4-2.

4.7.4 Finding A when Given P

Taking Equation (4-10) and solving for A, we find that

$$A = P\left[\frac{i(1+i)^N}{(1+i)^N - 1}\right]. \tag{4-14}$$

Thus, Equation (4-14) is the relation for finding the amount, A, of a uniform series of cash flows occurring at the end of each of N interest periods that would be equivalent to, or could be traded for, the present equivalent P, occurring at the beginning of the first period. The quantity in brackets is called the *capital recovery factor*.* Numerical values for this factor are given in the seventh column of the tables in Appendix C for a wide range of values of i and N. We shall use the functional symbol $(A/P, i\%, N)$ for this factor. Hence,

$$A = P(A/P, i\%, N). \tag{4-15}$$

An example that uses the equivalence between a present lump-sum loan amount and a series of equal uniform monthly payments starting at the end of month one and continuing through month four was provided in Table 4-1 as Plan 2.

* The capital recovery factor is more conveniently expressed as $i/[1-(1+i)^{-N}]$ for computation with a hand-held calculator.

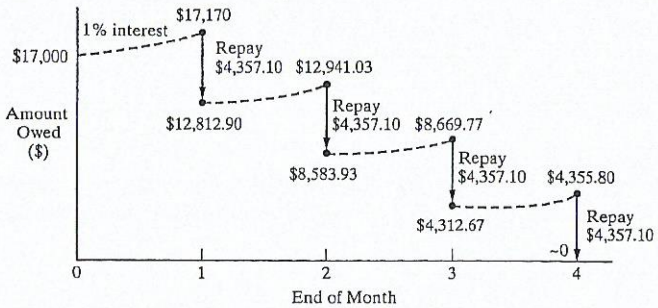


Figure 4-7 Relationship of Cash Flows for Plan 2 of Table 4-1 to Repayment of the $17,000 Loan Principal

Equation (4-15) yields the equivalent value of A that repays the $17,000 loan plus 1% interest per month over four months:

$$A = \$17{,}000(A/P, 1\%, 4) = \$17{,}000(0.2563) = \$4{,}357.10$$

The entries in columns three and five of Plan 2 in Table 4-1 can now be better understood. Interest owed at the end of month one equals $17,000(0.01), and therefore the principal repaid out of the total end-of-month payment of $4,357.10 is the difference, $4,187.10. At the beginning of month two, the amount of principal owed is $17,000 − $4,187.10 = $12,812.90. Interest owed at the end of month two is $12,812.90(0.01) = $128.13, and the principal repaid at that time is $4,357.10 − $128.13 = $4,228.97. The remaining entries in Plan 2 are obtained by performing these calculations for months three and four.

A graphical summary of Plan 2 is given in Figure 4-7. Here it can be seen that 1% interest is being paid on the beginning-of-month amount owed and that month-end payments of $4,357.10, consisting of interest and principal, bring the amount owed to $0 at the end of the fourth month. (The exact value of A is $4,356.78 and produces an exact value of $0 at the end of four months.) It is important to note that all the uniform series interest factors in Table 4-2 involve the same concept as the one illustrated in Figure 4-7.

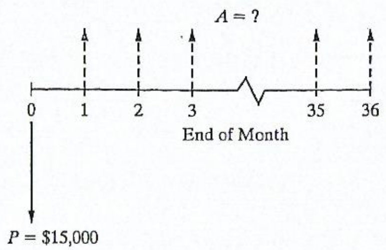
EXAMPLE 4-11 **Computing Your Monthly Car Payment**

You borrow $15,000 from your credit union to purchase a used car. The interest rate on your loan is 0.25% per month* and you will make a total of 36 monthly payments. What is your monthly payment?

* A good credit score (rating) can help you secure lower interest rates on loans. The Web site www.annualcreditreport.com allows you to check your credit score once per year at no cost.

Solution

The cash-flow diagram shown below is drawn from the viewpoint of the bank. Notice that the present amount of $15,000 occurs one month (interest period) *before* the first cash flow of the uniform repayment series.



The amount of the car payment is easily calculated using Equation (4-15).

$$A = \$15{,}000(A/P, 1/4\%, 36)$$
$$= \$15{,}000(0.0291)$$
$$= \$436.50 \text{ per month}$$

Another example of a problem where we desire to compute an equivalent value for A, from a given value of P and a known interest rate and number of compounding periods, is given in Table 4-2.

For an annual interest rate of 10%, the reader should now be convinced from Table 4-2 that $1,000 at the beginning of year one is equivalent to $187.45 at the EOYs one through eight, which is then equivalent to $2,143.60 at EOY eight.

4.7.5 Finding the Number of Cash Flows in an Annuity Given *A*, *P*, and *i*

Sometimes we may have information about a present amount of money (P), the magnitude of an annuity (A), and the interest rate (i). The unknown factor in this case is the number of cash flows in the annuity (N). Although there is not a closed-form equation for finding N, we can use the relationship between P and A to determine N.

EXAMPLE 4-12 Prepaying a Loan—Finding *N*

Your company has a $100,000 loan for a new security system it just bought. The annual payment is $8,880 and the interest rate is 8% per year for 30 years. Your company decides that it can afford to pay $10,000 per year. After how many payments (years) will the loan be paid off?

Solution

The original loan payment was found using Equation (4-15).

$$A = \$100{,}000\ (A/P, 8\%, 30) = \$100{,}000\ (0.0888) = \$8{,}800 \text{ per year}$$

Now, instead of paying $8,880 per year, your company is going to pay $10,000 per year. Common sense tells us that less than 30 payments will be necessary to pay off the $100,000 loan. Using Equation (4-11), we find

$$\$100{,}000 = \$10{,}000\ (P/A, 8\%, N)$$
$$(P/A, 8\%, N) = 10.$$

We can now use the interest tables provided in Appendix C to find N. Looking down the Present Worth Factor column (P/A) of Table C-11, we see that

$$(P/A,\ 8\%,\ 20) = 9.8181$$

and

$$(P/A,\ 8\%,\ 21) = 10.0168.$$

So, if $10,000 is paid per year, the loan will be paid off after 21 years instead of 30. The exact amount of the 21st payment will be slightly less than $10,000 (but we'll save that solution for another example).

Spreadsheet Solution

There is a financial function in Excel that would allow us to solve for the unknown number of periods. NPER(*rate*, *pmt*, *pv*) will compute the number of payments of magnitude *pmt* required to pay off a present amount (*pv*) at a fixed interest rate (*rate*).

$$N = \text{NPER}(0.08, -10000, 100000) = 20.91$$

Note that from your company's viewpoint, it received $100,000 (a cash inflow) at time 0 and is making $10,000 payments (cash outflows). Hence the annuity is expressed as a negative number in NPER( ) and the present amount as a positive number. If we were to reverse the signs—which would represent the lender's viewpoint—the same result would be obtained, namely NPER(0.08, 10000, −100000) = 20.91.

Comment

Prepaying loans can save thousands of dollars in interest. For example, look at the total interest paid under these two repayment plans.

Original payment plan ($8,880 per year for 30 years):

$$\text{Total interest paid} = \$8{,}880 \times 30 - \$100{,}000 = \$166{,}400$$

New payment plan ($10,000 per year for 21 years):

$$\text{Total interest paid} = \$10{,}000 \times 21 - \$100{,}000 = \$110{,}000$$

Prepaying the loan in this way would save $56,400 in interest!

4.7.6 Finding the Interest Rate, i, Given A, F, and N

Now let's look at the situation in which you know the amount (A) and duration (N) of a uniform payment series. You also know the desired future value of the series (F). What you don't know is the interest rate that makes them equivalent. As was the case for an unknown N, there is no single equation to determine i. However, we can use the known relationships between i, A, F, and N and the method of linear interpolation to approximate the interest rate.

EXAMPLE 4-13 **Finding the Interest Rate to Meet an Investment Goal**

After years of being a poor, debt-encumbered college student, you decide that you want to pay for your dream car in cash. Not having enough money now, you decide to specifically put money away each year in a "dream car" fund. The car you want to buy will cost $60,000 in eight years. You are going to put aside $6,000 each year (for eight years) to save for this. At what interest rate must you invest your money to achieve your goal of having enough to purchase the car after eight years?

Solution

We can use Equation (4-9) to show our desired equivalence relationship.

$$\$60{,}000 = \$6{,}000\,(F/A,\ i\%,\ 8)$$

$$(F/A,\ i\%,\ 8) = 10$$

Now we can use the interest tables in Appendix C to help track down the unknown value of i. What we are looking for are two interest rates, one with an $(F/A,\ i\%,\ 8)$ value greater than 10 and one with an $(F/A,\ i\%,\ 8)$ less than 10. Thumbing through Appendix C, we find

$$(F/A,\ 6\%,\ 8) = 9.8975 \quad \text{and} \quad (F/A,\ 7\%,\ 8) = 10.2598,$$

which tells us that the interest rate we are looking for is between 6% and 7% per year. Even though the function $(F/A, i\%, N)$ is nonlinear, we can use linear interpolation to approximate the value of i.

The dashed curve in Figure 4-8 is what we are linearly approximating. The answer, i', can be determined by using the similar triangles dashed in Figure 4-8.

$$\frac{\text{line dA}}{\text{line ed}} = \frac{\text{line BA}}{\text{line CB}}$$

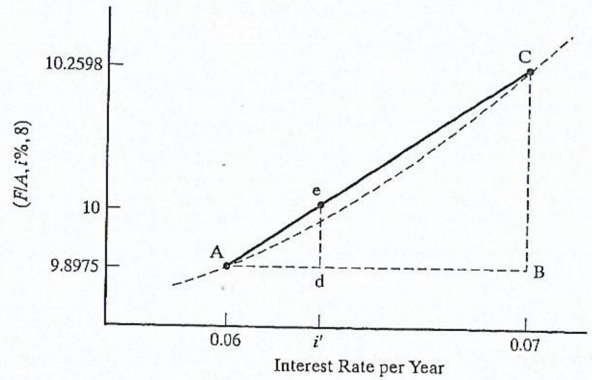


Figure 4-8 Using Linear Interpolation to Approximate i in Example 4-13

$$\frac{i' - 0.06}{10 - 9.8975} = \frac{0.07 - 0.06}{10.2598 - 9.8975}$$

$$i' = 0.0628 \text{ or } 6.28\% \text{ per year}$$

So if you can find an investment account that will earn at least 6.28% interest per year, you'll have the $60,000 you need to buy your dream car in eight years.

Spreadsheet Solution

Excel has another financial function that allows you to solve for an unknown interest rate. RATE(*nper*, *pmt*, *pv*, *fv*) will return the fixed interest rate that equates an annuity of magnitude *pmt* that lasts for *nper* periods to either its present value (*pv*) or its future value (*fv*).

$$i' = \text{RATE}(8, -6000, 0, 60000) = 0.0629 \text{ or } 6.29\% \text{ per year}$$

Note that a 0 was entered for *pv* since we were working with a known future value in this example.

4.8 Summary of Interest Formulas and Relationships for Discrete Compounding

Table 4-3 provides a summary of the six most common discrete compound interest factors, utilizing notation of the preceding sections. The formulas are for *discrete compounding*, which means that the interest is compounded at the end of each finite-length period, such as a month or a year. Furthermore, the formulas also assume discrete (i.e., lump sum) cash flows spaced at the end of equal time intervals on a cash-flow diagram. Discrete compound interest factors are given in Appendix C, where the assumption is made that i remains constant during the N compounding periods.

There are also several useful relationships between the compound interest factors. These relationships are summarized in the following equations.

$$(P/F, i\%, N) = \frac{1}{(F/P, i\%, N)}; \tag{4-16}$$

$$(A/P, i\%, N) = \frac{1}{(P/A, i\%, N)}; \tag{4-17}$$

$$(A/F, i\%, N) = \frac{1}{(F/A, i\%, N)}; \tag{4-18}$$

$$(F/A, i\%, N) = (P/A, i\%, N)(F/P, i\%, N); \tag{4-19}$$

$$(P/A, i\%, N) = \sum_{k=1}^{N} (P/F, i\%, k); \tag{4-20}$$

TABLE [illegible] Discrete Compounding-Interest Factors and Symbols[a]

To Find:	Given:	Factor by which to Multiply "Given"[a]	Factor Name	Factor Functional Symbol[b]
For single cash flows:				
F	P	$(1+i)^N$	Single payment compound amount	$(F/P, i\%, N)$
P	F	$\frac{1}{(1+i)^N}$	Single payment present worth	$(P/F, i\%, N)$
For uniform series (annuities):				
F	A	$\frac{(1+i)^N - 1}{i}$	Uniform series compound amount	$(F/A, i\%, N)$
P	A	$\frac{(1+i)^N - 1}{i(1+i)^N}$	Uniform series present worth	$(P/A, i\%, N)$
A	F	$\frac{i}{(1+i)^N - 1}$	Sinking fund	$(A/F, i\%, N)$
A	P	$\frac{i(1+i)^N}{(1+i)^N - 1}$	Capital recovery	$(A/P, i\%, N)$

[a] i equals effective interest rate per interest period; N, number of interest periods; A, uniform series amount (occurs at the end of each interest period); F, future equivalent; P, present equivalent.

[b] The functional symbol system is used throughout this book.

$$(F/A, i\%, N) = \sum_{k=1}^{N} (F/P, i\%, N-k); \tag{4-21}$$

$$(A/F, i\%, N) = (A/P, i\%, N) - i. \tag{4-22}$$

4.9 Deferred Annuities (Uniform Series)

All annuities (uniform series) discussed to this point involve the first cash flow being made at the end of the first period, and they are called *ordinary annuities*. If the cash flow does not begin until some later date, the annuity is known as a *deferred annuity*. If the annuity is deferred for J periods ($J < N$), the situation is as portrayed in Figure 4-9, in which the entire framed ordinary annuity has been moved away from "time present," or "time zero," by J periods. Remember that, in an annuity deferred for J periods, the first payment is made at the end of period $(J+1)$, assuming that all periods involved are equal in length.

The present equivalent at the end of period J of an annuity with cash flows of amount A is, from Equation (4-9), $A(P/A, i\%, N-J)$. The present equivalent of the single amount $A(P/A, i\%, N-J)$ as of time zero will then be

$$P_0 = A(P/A, i\%, N-J)(P/F, i\%, J).$$

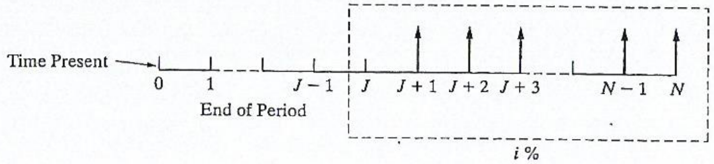


Figure 4-9 General Cash-Flow Representation of a Deferred Annuity (Uniform Series)

EXAMPLE 4-14 Present Equivalent of a Deferred Annuity

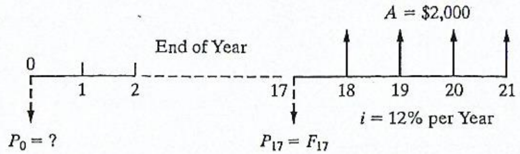
To illustrate the preceding discussion, suppose that a father, on the day his son is born, wishes to determine what lump amount would have to be paid into an account bearing interest of 12% per year to provide withdrawals of \$2,000 on each of the son's 18th, 19th, 20th, and 21st birthdays.

Solution

The problem is represented in the following cash-flow diagram. One should first recognize that an ordinary annuity of four withdrawals of \$2,000 each is involved and that the present equivalent of this annuity occurs at the 17th birthday when a $(P/A, i\%, N-J)$ factor is utilized. In this problem, $N = 21$ and $J = 17$. It is often helpful to use a *subscript* with P or F to denote the respective point in time.

Hence,

$$P_{17} = A(P/A, 12\%, 4) = \$2{,}000(3.0373) = \$6{,}074.60.$$



Note the dashed arrow in the cash-flow diagram denoting P_{17}. Now that P_{17} is known, the next step is to calculate P_0. With respect to P_0, P_{17} is a future equivalent, and hence it could also be denoted F_{17}. Money at a given point in time, such as the end of period 17, is the same regardless of whether it is called a present equivalent or a future equivalent. Hence,

$$P_0 = F_{17}(P/F, 12\%, 17) = \$6{,}074.60(0.1456) = \$884.46,$$

which is the amount that the father would have to deposit on the day his son is born.

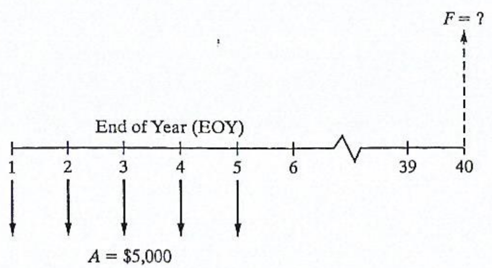
EXAMPLE 4-15

Deferred Future Value of an Annuity

When you take your first job, you decide to start saving right away for your retirement. You put \$5,000 per year into the company's 401(k) plan, which averages 8% interest per year. Five years later, you move to another job and start a new 401(k) plan. You never get around to merging the funds in the two plans. If the first plan continued to earn interest at the rate of 8% per year for 35 years after you stopped making contributions, how much is the account worth?

Solution

The following cash-flow diagram clarifies the timing of the cash flows for the original 401(k) investment plan.



The easiest way to approach this is to first find the future equivalent of the annuity as of time 5.

$$F_5 = \$5{,}000\,(F/A, 8\%, 5) = \$5{,}000\,(5.8666) = \$29{,}333.$$

To determine F_{40}, F_5 can now be denoted P_5, and

$$F_{40} = P_5\,(F/P, 8\%, 35) = \$29{,}333\,(14.7853) = \$433{,}697.$$

4.10 Equivalence Calculations Involving Multiple Interest Formulas

You should now be comfortable with equivalence problems that involve discrete compounding of interest and discrete cash flows. All compounding of interest takes place once per time period (e.g., a year), and to this point cash flows also occur once per time period. This section provides examples involving two or more equivalence calculations to solve for an unknown quantity. The end-of-year cash-flow convention is used. Again, the interest rate is constant over the N time periods.

EXAMPLE 4-16

Calculating Equivalent *P*, *F*, and *A* Values

Figure 4-10 depicts an example problem with a series of year-end cash flows extending over eight years. The amounts are \$100 for the first year, \$200 for the second year, \$500 for the third year, and \$400 for each year from the fourth through the eighth. These could represent something like the expected maintenance expenditures for a certain piece of equipment or payments into a fund. Note that the payments are shown at the end of each year, which is a standard assumption (convention) for this book and for economic analyses in general, unless we have information to the contrary. It is desired to find

(a) the present equivalent expenditure, P_0;
(b) the future equivalent expenditure, F_8;
(c) the annual equivalent expenditure, A

of these cash flows if the annual interest rate is 20%. Solve by hand and by using a spreadsheet.

Solution by Hand

(a) To find the equivalent P_0, we need to sum the equivalent values of all payments as of the beginning of the first year (time zero). The required movements of money through time are shown graphically in Figure 4-10(a).

$$
\begin{aligned}
P_0 = {} & F_1(P/F, 20\%, 1) && = \$100(0.8333) && = \$83.33 \\
& + F_2(P/F, 20\%, 2) && + \$200(0.6944) && + 138.88 \\
& + F_3(P/F, 20\%, 3) && + \$500(0.5787) && + 289.35 \\
& + A(P/A, 20\%, 5) \times (P/F, 20\%, 3) && + \$400(2.9900) \times (0.5787) && + 692.26 \\
& && && \$1{,}203.82.
\end{aligned}
$$

(b) To find the equivalent F_8, we can sum the equivalent values of all payments as of the end of the eighth year (time eight). Figure 4-10(b) indicates these

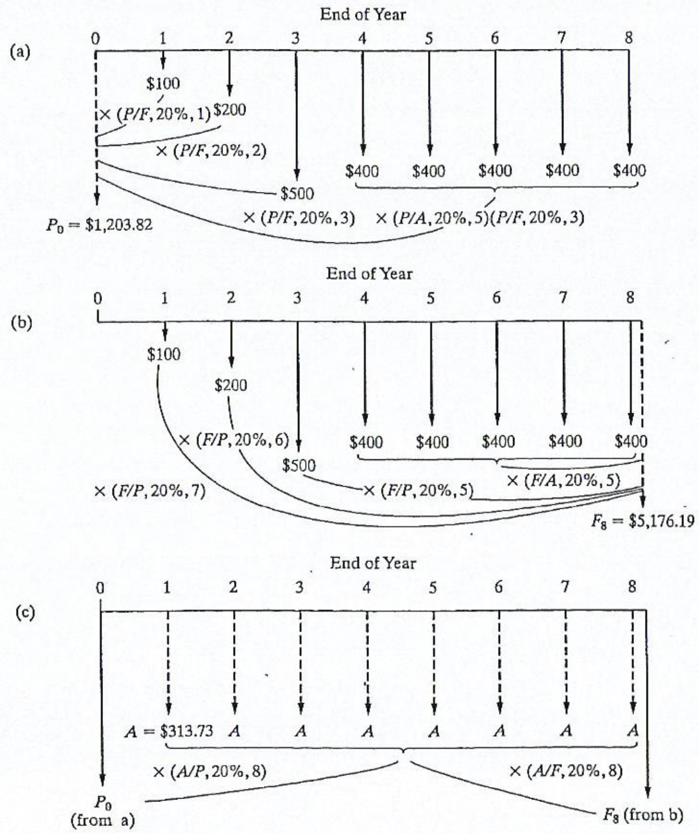


Figure 4-10 Example 4-16 for Calculating the Equivalent P, F, and A Values

movements of money through time. However, since the equivalent P_0 is already known to be $1,203.82, we can directly calculate

$$F_8 = P_0(F/P, 20\%, 8) = \$1{,}203.82(4.2998) = \$5{,}176.19.$$

(c) The equivalent A of the irregular cash flows can be calculated directly from either P_0 or F_8 as

$$A = P_0(A/P, 20\%, 8) = \$1{,}203.82(0.2606) = \$313.73$$

or

$$A = F_8(A/F, 20\%, 8) = \$5{,}176.19(0.0606) = \$313.73.$$

The computation of A from P_0 and F_8 is shown in Figure 4-10(c). Thus, we find that the irregular series of payments shown in Figure 4-10 is equivalent to $1,203.82 at time zero, $5,176.19 at time eight, or a uniform series of $313.73 at the end of each of the eight years.

Spreadsheet Solution

Figure 4-11 displays a spreadsheet solution for this example. The present equivalent (P_0) of the tabulated cash flows is easily computed by using the NPV function with the stated interest rate (20% in cell B1). The future equivalent (F_8) is determined from the present equivalent by using the $(F/P, i\%, N)$ relationship. The annual equivalent is also determined from the present equivalent by

Figure 4-11 Spreadsheet Solution, Example 4-16

	A	B	C	D	E
1	i / yr	20%		EOY	Cash Flow
2	Years in series	8		0	$ -
3				1	$ 100
4				2	$ 200
5				3	$ 500
6				4	$ 400
7				5	$ 400
8				6	$ 400
9				7	$ 400
10				8	$ 400
11					
12	P_0 =	$ 1,204			
13	F_8 =	$ 5,176			
14	A =	$ 314			

= NPV(B1, E3:E10)

= B12 * (1 + B1) ^ B2

= PMT(B1, B2, –B12)

applying the PMT function. The slight differences in results when compared to the hand solution are due to rounding of the interest factor values in the hand solution.

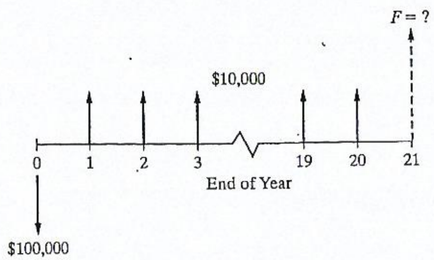
EXAMPLE 4-17 How Much Is that Last Payment? (Example 4-12 Revisited)

In Example 4-12, we looked at paying off a loan early by increasing the annual payment. The \$100,000 loan was to be repaid in 30 annual installments of \$8,880 at an interest rate of 8% per year. As part of the example, we determined that the loan could be paid in full after 21 years if the annual payment was increased to \$10,000.

As with most real-life loans, the final payment will be something different (usually less) than the annuity amount. This is due to the effect of rounding in the interest calculations—you can't pay in fractions of a cent! For this example, determine the amount of the 21st (and final) payment on the \$100,000 loan when 20 payments of \$10,000 have already been made. The interest rate remains at 8% per year.

Solution

The cash-flow diagram for this example is shown below. It is drawn from the lender's viewpoint.



We need to determine the value of F that will make the present equivalent of all loan payments equal to the amount borrowed. We can do this by discounting all of the payments to time 0 (including the final payment, F) and setting their value equal to \$100,000.

$$\$10{,}000\,(P/A, 8\%, 20) + F\,(P/F, 8\%, 21) = \$100{,}000$$
$$\$10{,}000\,(9.8181) + F\,(0.1987) = \$100{,}000$$
$$F = \$9{,}154.50$$

Thus, a payment of \$9,154.50 is needed at the end of year 21 to pay off the loan.

EXAMPLE 4-18 The Present Equivalent of BP's Payment Schedule

In this example, we answer the question posed at the beginning of the chapter—what is the present equivalent value of BP's proposed payment schedule? Recall that BP will pay \$3 billion at the end of the third quarter of 2010 and another \$2 billion in the fourth quarter of 2010. Twelve additional payments of \$1.25 billion each quarter thereafter will result in a total of \$20 billion having been paid into the fund. The interest rate is 3% per quarter.

Solution

Figure 4-12 shows the cash-flow diagram for this situation. The present equivalent of the cash flows is

$$P = \$3 \text{ billion } (P/F, 3\%, 1) + \$2 \text{ billion } (P/F, 3\%, 2)$$
$$+ \$1.25 \text{ billion } (P/A, 3\%, 12)\,(P/F, 3\%, 2)$$
$$= \$3 \text{ billion } (0.9709) + \$2 \text{ billion } (0.9426) + \$1.25\,(9.9540)(0.9426)$$
$$= \$16.53 \text{ billion.}$$

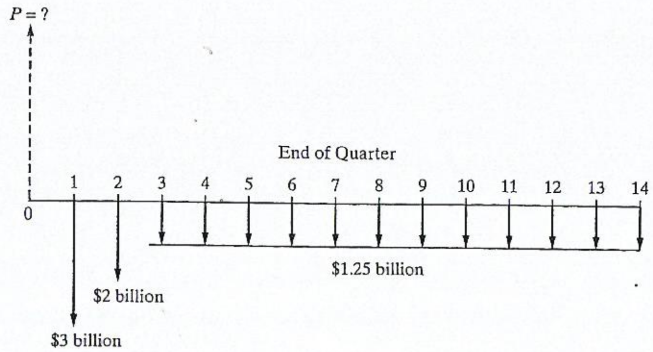


Figure 4-12 Cash-Flow Diagram for Example 4-18

EXAMPLE 4-19 Determining an Unknown Annuity Amount

Two receipts of \$1,000 each are desired at the EOYs 10 and 11. To make these receipts possible, four EOY annuity amounts will be deposited in a bank at EOYs 2, 3, 4, and 5. The bank's interest rate (i) is 12% per year.

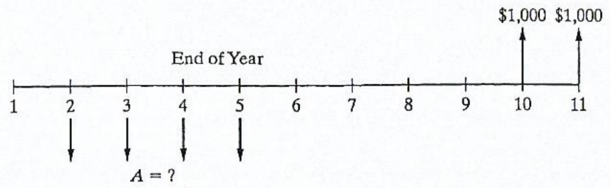
(a) Draw a cash-flow diagram for this situation.

(b) Determine the value of A that establishes equivalence in your cash-flow diagram.

(c) Determine the lump-sum value at the end of year 11 of the completed cash-flow diagram based on your answers to Parts (a) and (b).

Solution

(a) Cash-flow diagrams can make a seemingly complex problem much clearer. The cash-flow diagram for this example is shown below.



(b) Because the unknown annuity, A, begins at EOY two, it makes sense to establish our reference year for the equivalence calculations at EOY one (remember that the first annuity amount follows its P-equivalent amount by one year). So the P-equivalent at EOY 1 of the four A amounts is

$$P_1 = A(P/A, 12\%, 4).$$

Next we calculate the EOY one P-equivalent of \$1,000 at EOY 10 and \$1,000 at EOY 11 as follows:

$$P_1' = \$1{,}000\,(P/A, 12\%, 2)\,(P/F, 12\%, 8).$$

The $(P/F, 12\%, 8)$ factor is needed to discount the equivalent value of the A amounts at EOY nine to EOY one. By equating both P-equivalents at EOY one, we can solve for the unknown amount, A.

$$P_1 = P_1'$$

$$A\,(P/A, 12\%, 4) = \$1{,}000\,(P/A, 12\%, 2)\,(P/F, 12\%, 8),$$

or

$$3.0373\,A = \$682.63$$

and

$$A = \$224.75.$$

Therefore, we conclude that deposits of \$224.75 at EOYs two, three, four, and five are equivalent to \$1,000 at EOYs 10 and 11 if the interest rate is 12% per year.

(c) Now we need to calculate the F-equivalent at time 11 of the −\$224.75 annuity in years 2 through 5 and the \$1,000 annuity in years 10 and 11.

$$-\$224.75\,(F/A, 12\%, 4)\,(F/P, 12\%, 6) + \$1{,}000\,(F/A, 12\%, 2)$$

$$= -\$0.15$$

This value should be zero, but round-off error in the interest factors causes a small difference of \$0.15.

4.11 Uniform (Arithmetic) Gradient of Cash Flows

Some problems involve receipts or expenses that are projected to increase or decrease by a uniform *amount* each period, thus constituting an arithmetic sequence of cash flows. For example, because of leasing a certain type of equipment, maintenance and repair savings relative to purchasing the equipment may increase by a roughly constant amount each period. This situation can be modeled as a *uniform gradient* of cash flows.

Figure 4-13 is a cash-flow diagram of a sequence of end-of-period cash flows increasing by a constant amount, G, in each period. The G is known as the *uniform gradient amount*. Note that the timing of cash flows on which the derived formulas and tabled values are based is as follows:

End of Period	Cash Flows
1	$(0)G$
2	$(1)G$
3	$(2)G$
.	.
.	.
.	.
$N-1$	$(N-2)G$
N	$(N-1)G$

Notice that the first uniform gradient cash flow, G, occurs at the end of period two.

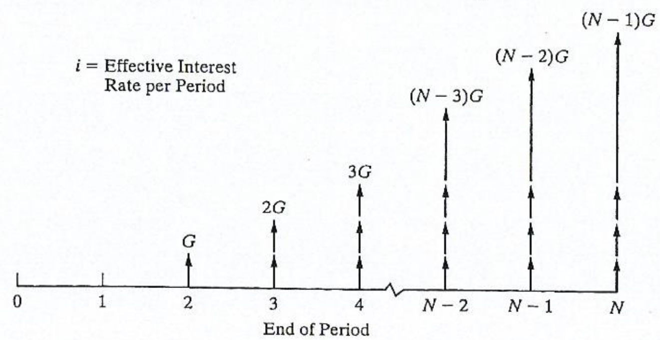


Figure 4-13 Cash-Flow Diagram for a Uniform Gradient Increasing by G Dollars per Period

4.11.1 Finding P when Given G

The present equivalent, P, of the arithmetic sequence of cash flows shown in Figure 4-13 is

$$P = \frac{G(1)}{(1+i)^2} + \frac{G(2)}{(1+i)^3} + \frac{G(3)}{(1+i)^4} + \cdots + \frac{G(N-2)}{(1+i)^{N-1}} + \frac{G(N-1)}{(1+i)^N}.$$

If we add in the dummy term $G(0)/(1+i)^1$ to represent the "missing" cash flow at time one, we can rewrite the above equation as:

$$P = G\sum_{n=1}^{N} \frac{(n-1)}{(1+i)^n}.$$

Recognizing the above equation as the summation of a geometric sequence, we can make the appropriate substitutions as we did in the development of Equation (4-6). After some algebraic manipulation, we have

$$P = G\left\{\frac{1}{i}\left[\frac{(1+i)^N - 1}{i(1+i)^N} - \frac{N}{(1+i)^N}\right]\right\}. \qquad (4\text{-}23)$$

The term in braces in Equation (4-23) is called the *gradient to present equivalent conversion factor*. It can also be expressed as $(1/i)[(P/A, i\%, N) - N(P/F, i\%, N)]$. Numerical values for this factor are given in column 8 of Appendix C for a wide assortment of i and N values. We shall use the functional symbol $(P/G, i\%, N)$ for this factor. Hence,

$$P = G\,(P/G,\, i\%,\, N). \qquad (4\text{-}24)$$

4.11.2 Finding A when Given G

From Equation (4-23), it is easy to develop an expression for A as follows:

$$\begin{aligned}
A &= P(A/P, i\%, N) \\
&= G\left\{\frac{1}{i}\left[\frac{(1+i)^N - 1}{i(1+i)^N} - \frac{N}{(1+i)^N}\right]\right\}(A/P, i\%, N) \\
&= \frac{G}{i}\left[(P/A, i\%, N) - \frac{N}{(1+i)^N}\right](A/P, i\%, N) \\
&= \frac{G}{i}\left[1 - \frac{Ni(1+i)^N}{(1+i)^N\left[(1+i)^N - 1\right]}\right] \\
&= \frac{G}{i} - G\left[\frac{N}{(1+i)^N - 1}\right] \\
&= G\left[\frac{1}{i} - \frac{N}{(1+i)^N - 1}\right]. \qquad (4\text{-}25)
\end{aligned}$$

The term in brackets in Equation (4-25) is called the *gradient to uniform series conversion factor*. Numerical values for this factor are given on the right-hand side of Appendix C for a range of i and N values. We shall use the functional symbol $(A/G, i\%, N)$ for this factor. Thus,

$$A = G\,(A/G,\, i\%,\, N). \qquad (4\text{-}26)$$

4.11.3 Finding F when Given G

We can develop an equation for the future equivalent, F, of an arithmetic series using Equation (4-23):

$$\begin{aligned}
F &= P(F/P,\, i\%,\, N) \\
&= G\left\{\frac{1}{i}\left[\frac{(1+i)^N - 1}{i(1+i)^N} - \frac{N}{(1+i)^N}\right]\right\}(1+i)^N \\
&= G\left\{\frac{1}{i}\left[\frac{(1+i)^N - 1}{i} - N\right]\right\} \\
&= \frac{G}{i}(F/A,\, i\%,\, N) - \frac{NG}{i}. \qquad (4\text{-}27)
\end{aligned}$$

It is usually more practical to deal with present and annual equivalents of arithmetic series.

4.11.4 Computations Using G

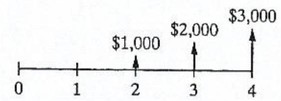
Be sure to notice that the direct use of gradient conversion factors applies when there is no cash flow at the end of period one, as in Example 4-20. There may be an A amount at the end of period one, but it is treated separately, as illustrated in Examples 4-21 and 4-22. A major advantage of using gradient conversion factors (i.e., computational time savings) is realized when N becomes large.

EXAMPLE 4-20 Using the Gradient Conversion Factors to Find P and A

As an example of the straightforward use of the gradient conversion factors, suppose that certain EOY cash flows are expected to be \$1,000 for the *second* year, \$2,000 for the third year, and \$3,000 for the fourth year and that, if interest is 15% per year, it is desired to find

(a) present equivalent value at the beginning of the first year,
(b) uniform annual equivalent value at the end of each of the four years.

Solution



Observe that this schedule of cash flows fits the model of the arithmetic gradient formulas with $G = \$1{,}000$ and $N = 4$. Note that there is no cash flow at the end of the first period.

(a) The present equivalent can be calculated as

$$P_0 = G(P/G, 15\%, 4) = \$1{,}000(3.79) = \$3{,}790.$$

(b) The annual equivalent can be calculated from Equation (4-26) as

$$A = G(A/G, 15\%, 4) = \$1{,}000(1.3263) = \$1{,}326.30.$$

Of course, once P_0 is known, the value of A can be calculated as

$$A = P_0(A/P, 15\%, 4) = \$3{,}790(0.3503) = \$1{,}326.30.$$

EXAMPLE 4-21

Present Equivalent of an Increasing Arithmetic Gradient Series

As a further example of the use of arithmetic gradient formulas, suppose that we have cash flows as follows:

End of Year	Cash Flows ($)
1	5,000
2	6,000
3	7,000
4	8,000

Also, assume that we wish to calculate their present equivalent at $i = 15\%$ per year, using gradient conversion factors.

Solution

The schedule of cash flows is depicted in the left-hand diagram of Figure 4-14. The right two diagrams of Figure 4-14 show how the original schedule can be broken into two separate sets of cash flows, an annuity series of $5,000 plus an arithmetic gradient of $1,000 that fits the general gradient model for which factors are tabled. The summed present equivalents of these two separate sets of cash flows equal the present equivalent of the original problem. Thus, using the symbols shown in Figure 4-14, we have

$$P_{0T} = P_{0A} + P_{0G}$$

$$= A(P/A, 15\%, 4) + G(P/G, 15\%, 4)$$

$$= \$5{,}000(2.8550) + \$1{,}000(3.79) = \$14{,}275 + 3{,}790 = \$18{,}065.$$

The annual equivalent of the original cash flows could be calculated with the aid of Equation (4-26) as follows:

$$A_T = A + A_G$$

$$= \$5{,}000 + \$1{,}000(A/G,\ 15\%,\ 4) = \$6{,}326.30.$$

A_T is equivalent to P_{0T} because $6,326.30$(P/A, 15\%, 4) = \$18{,}061$, which is the same value obtained previously (subject to round-off error).

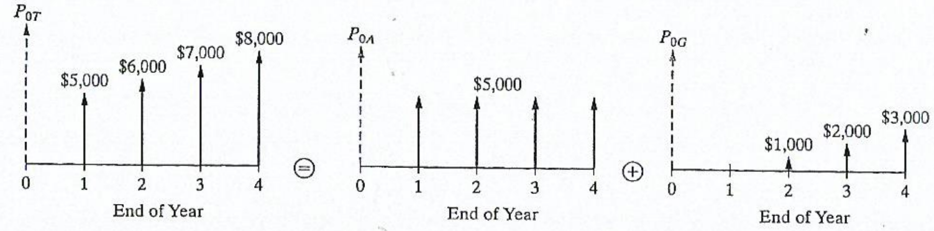


Figure 4-14 Breakdown of Cash Flows for Example 4-21

EXAMPLE 4-22

Present Equivalent of a Decreasing Arithmetic Gradient Series

For another example of the use of arithmetic gradient formulas, suppose that we have cash flows that are timed in exact reverse of the situation depicted in Example 4-21. The left-hand diagram of Figure 4-15 shows the following sequence of cash flows:

End of Year	Cash Flows ($)
1	8,000
2	7,000
3	6,000
4	5,000

Calculate the present equivalent at $i = 15\%$ per year, using arithmetic gradient interest factors.

Solution

The right two diagrams of Figure 4-15 show how the uniform gradient can be broken into two separate cash-flow diagrams. In this example, we are *subtracting* an arithmetic gradient of $1,000 from an annuity series of $8,000.

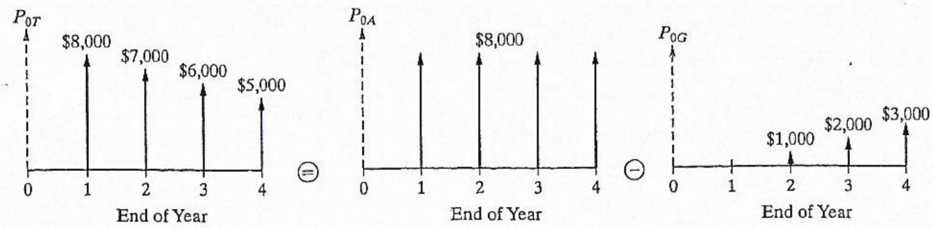


Figure 4-15 Breakdown of Cash Flows for Example 4-22

So,

$$
\begin{aligned}
P_{0T} &= P_{0A} - P_{0G} \\
&= A(P/A, 15\%, 4) - G(P/G, 15\%, 4) \\
&= \$8{,}000(2.8550) - \$1{,}000(3.79) \\
&= \$22{,}840 - \$3{,}790 = \$19{,}050.
\end{aligned}
$$

Again, the annual equivalent of the original decreasing series of cash flows can be calculated by the same rationale:

$$
\begin{aligned}
A_T &= A - A_G \\
&= \$8{,}000 - \$1{,}000(A/G, 15\%, 4) \\
&= \$6{,}673.70.
\end{aligned}
$$

Note from Examples 4-21 and 4-22 that the present equivalent of $18,065 for an increasing arithmetic gradient series of payments is different from the present equivalent of $19,050 for an arithmetic gradient of payments of identical amounts, but with reversed timing (decreasing series of payments). This difference would be even greater for higher interest rates and gradient amounts and exemplifies the marked effect of the timing of cash flows on equivalent values.

4.12 Geometric Sequences of Cash Flows

Some economic equivalence problems involve projected cash-flow patterns that are changing at an average *rate*, $\bar{f}$, each period. A fixed amount of a commodity that inflates in price at a constant rate each year is a typical situation that can be modeled with a geometric sequence of cash flows. The resultant EOY cash-flow pattern is referred to as a *geometric gradient series* and has the general appearance shown in Figure 4-16. Notice that *the initial cash flow in this series*, A_1, *occurs at the end of period* one and that $A_k = (A_{k-1})(1+\bar{f})$, $2 \le k \le N$. The Nth term in

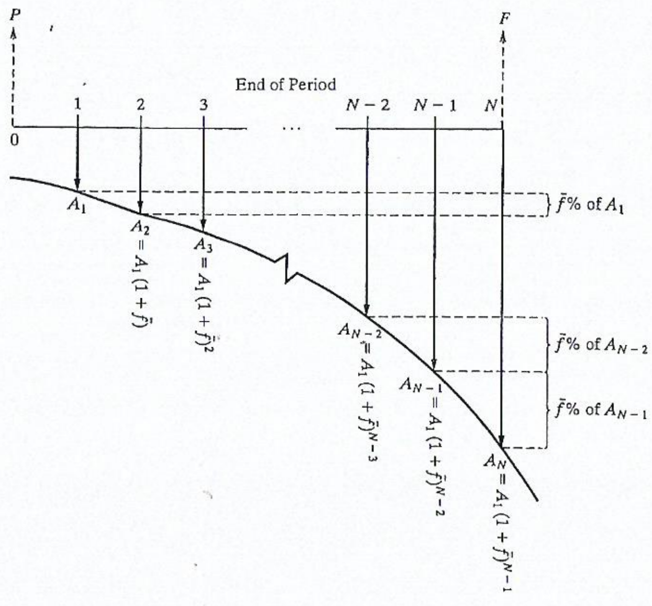


Figure 4-16 Cash-Flow Diagram for a Geometric Sequence of Payments Increasing at a Constant Rate of $\bar{f}$ per Period

this geometric sequence is $A_N = A_1(1+\bar{f})^{N-1}$, and the common ratio throughout the sequence is $(A_k - A_{k-1})/A_{k-1} = \bar{f}$. Be sure to notice that $\bar{f}$ can be positive *or* negative.

Each term in Figure 4-16 could be discounted, or compounded, at interest rate i per period to obtain a value of P or F, respectively. However, this becomes quite tedious for large N, so it is convenient to have a single equation instead.

The present equivalent of the geometric gradient series shown in Figure 4-16 is

$$
\begin{aligned}
P &= A_1(P/F, i\%, 1) + A_2(P/F, i\%, 2) + A_3(P/F, i\%, 3) \\
&\quad + \cdots + A_N(P/F, i\%, N) \\
&= A_1(1+i)^{-1} + A_2(1+i)^{-2} + A_3(1+i)^{-3} + \cdots + A_N(1+i)^{-N} \\
&= A_1(1+i)^{-1} + A_1(1+\bar{f})(1+i)^{-2} + A_1(1+\bar{f})^2(1+i)^{-3} \\
&\quad + \cdots + A_1(1+\bar{f})^{N-1}(1+i)^{-N} \\
&= A_1(1+i)^{-1}[1 + x + x^2 + \cdots + x^{N-1}], \qquad (4\text{-}28)
\end{aligned}
$$

where $x = (1+\bar{f})/(1+i)$. The expression in brackets in Equation (4-28) reduces to $(1+x^N)/(1-x)$ when $x \ne 1$ or $\bar{f} \ne i$. If $\bar{f} = i$, then $x = 1$ and the expression in brackets reduces to N, the number of terms in the summation. Hence,

$$P = \begin{cases} A_1(1+i)^{-1}(1-x^N)/(1-x) & \bar{f} \neq i \\ A_1 N(1+i)^{-1} & \bar{f} = i, \end{cases}$$

which reduces to

$$P = \begin{cases} \dfrac{A_1[1-(1+i)^{-N}(1+\bar{f})^N]}{i-\bar{f}} & \bar{f} \neq i \\ A_1 N(1+i)^{-1} & \bar{f} = i, \end{cases} \quad (4\text{-}29)$$

or

$$P = \begin{cases} \dfrac{A_1[1-(P/F, i\%, N)(F/P, \bar{f}\%, N)]^*}{i-\bar{f}} & \bar{f} \neq i \\ A_1 N(P/F, i\%, 1) & \bar{f} = i. \end{cases} \quad (4\text{-}30)$$

Once we know the present equivalent of a geometric gradient series, we can easily compute the equivalent uniform series or future amount using the basic interest factors $(A/P, i\%, N)$ and $(F/P, i\%, N)$.

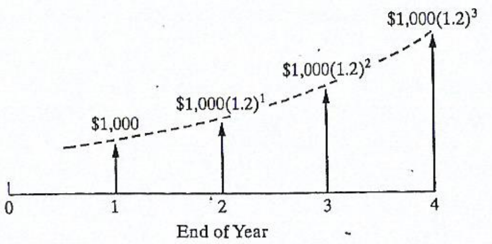
Additional discussion of geometric sequences of cash flows is provided in Chapter 8, which deals with price changes and exchange rates.

EXAMPLE 4-23

Equivalence Calculations for an Increasing Geometric Gradient Series

Consider the following EOY geometric sequence of cash flows and determine the P, A, and F equivalent values. The rate of increase is 20% per year after the first year, and the interest rate is 25% per year.

Solution



$$P = \frac{\$1{,}000\,[1-(P/F, 25\%, 4)(F/P, 20\%, 4)]}{0.25-0.20}$$

* Equation (4-30) for $\bar{f} \neq i$ is mathematically equivalent to the following:

$$P = \frac{A_1}{(1+\bar{f})}\left(P/A, \frac{1+i}{1+\bar{f}} - 1, N\right).$$

$$\begin{aligned} P &= \frac{\$1{,}000}{0.05}\left[1-(0.4096)(2.0736)\right] \\ &= \$20{,}000(0.15065) \\ &= \$3{,}013; \\ A &= \$3{,}013(A/P, 25\%, 4) = \$1{,}275.70; \\ F &= \$3{,}013(F/P, 25\%, 4) = \$7{,}355.94. \end{aligned}$$

EXAMPLE 4-24

Equivalence Calculations for a Decreasing Geometric Gradient Series

Suppose that the geometric gradient in Example 4-23 begins with \$1,000 at EOY one and *decreases* by 20% per year after the first year. Determine P, A, and F under this condition.

Solution

The value of $\bar{f}$ is −20% in this case. The desired quantities are as follows:

$$\begin{aligned} P &= \frac{\$1{,}000[1-(P/F, 25\%, 4)(F/P, -20\%, 4)]}{0.25-(-0.20)} \\ &= \frac{\$1{,}000}{0.45}\left[1-(0.4096)(1-0.20)^4\right] \\ &= \$2{,}222.22(0.83222) \\ &= \$1{,}849.38; \\ A &= \$1{,}849.38(A/P, 25\%, 4) = \$783.03; \\ F &= \$1{,}849.38(F/P, 25\%, 4) = \$4{,}515.08. \end{aligned}$$

EXAMPLE 4-25

Using a Spreadsheet to Model Geometric Gradient Series

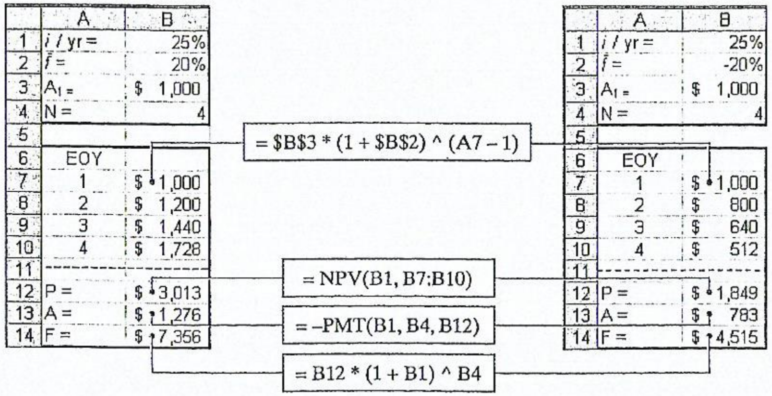
Create a spreadsheet to solve

(a) Example 4-23

(b) Example 4-24.

Solution

Figure 4-17 displays the spreadsheet solution for this problem. A formula is used to compute the actual cash flow for each year on the basis of EOY one cash flow (A_1) and the yearly rate of change ($\bar{f}$). Once we have the set of EOY cash flows, P can be computed by using the NPV function. The values of A and F are easily



(a) Solution to Example 4-23 (b) Solution to Example 4-24

Figure 4-17 Spreadsheet Solution, Example 4-25

derived from the value of P. Note that the structure of the spreadsheets for Parts (a) and (b) are the same—the only difference is the value of $\bar{f}$ in cell B2.

EXAMPLE 4-26

A Retirement Savings Plan

On your 23rd birthday you decide to invest \$4,500 (10% of your annual salary) in a mutual fund earning 7% per year. You will continue to make annual deposits equal to 10% of your annual salary until you retire at age 62 (40 years after you started your job). You expect your salary to increase by an average of 4% each year during this time. How much money will you have accumulated in your mutual fund when you retire?

Solution

Since the amount of your deposit is 10% of your salary, each year the amount you deposit will increase by 4% as your salary increases. Thus, your deposits constitute a geometric gradient series with $f = 4\%$ per year. We can use Equation (4-30) to determine the present equivalent amount of the deposits.

$$P = \frac{\$4{,}500[1 - (P/F, 7\%, 40)(F/P, 4\%, 40)]}{0.07 - 0.04}$$

$$= \frac{\$4{,}500[1 - (0.0668)(4.8010)]}{0.03}$$

$$= \$101{,}894.$$

Now the future worth at age 62 can be determined.

$$F = \$101{,}894(F/P, 7\%, 40)$$
$$= \$101{,}894(14.9745)$$
$$= \$1{,}525{,}812.$$

This savings plan will make you a millionaire when you retire. *Moral:* Start saving early!

4.13 Interest Rates that Vary with Time

Student loans under the U.S. government's popular Stafford program let students borrow money up to a certain amount each year (based on their year in school and financial need). Stafford loans are the most common type of educational loan, and they have a floating interest rate that readjusts every year (but cannot exceed 8.25% per year). When the interest rate on a loan can vary with time, it is necessary to take this into account when determining the future equivalent value of the loan. Example 4-27 demonstrates how this situation is treated.

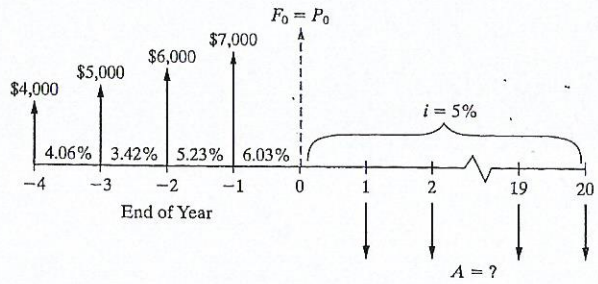
EXAMPLE 4-27

Compounding with Changing Interest Rates

Ashea Smith is a 22-year-old senior who used the Stafford loan program to borrow \$4,000 four years ago when the interest rate was 4.06% per year. \$5,000 was borrowed three years ago at 3.42%. Two years ago she borrowed \$6,000 at 5.23%, and last year \$7,000 was borrowed at 6.03% per year. Now she would like to consolidate her debt into a single 20-year loan with a 5% fixed annual interest rate. If Ashea makes annual payments (starting in one year) to repay her total debt, what is the amount of each payment?

Solution

The following cash-flow diagram clarifies the timing of Ashea's loans and the applicable interest rates. The diagram is drawn using Ashea's viewpoint.



Before we can find the annual repayment amount, we need to find the current (time 0) equivalent value of the four loans. This problem can be solved by compounding the amount owed at the beginning of each year by the interest rate that applies to each individual year and repeating this process over the four years to obtain the total current equivalent value.

$$F_{-3} = \$4{,}000(F/P, 4.06\%, 1) + \$5{,}000 = \$4{,}000(1.0406)$$
$$+ \$5{,}000 = \$9{,}162.40$$
$$F_{-2} = \$9{,}162.40(F/P, 3.42\%, 1) + \$6{,}000 = \$15{,}475.75$$
$$F_{-1} = \$15{,}475.75(F/P, 5.23\%, 1) + \$7{,}000 = \$23{,}285.13$$
$$F_0 = \$23{,}285.13(F/P, 6.03\%, 1) = \$24{,}689.22$$

Notice that it was a simple matter to substitute $(F/P, i\%, N) = (1 + i)^N$ for the noninteger values of i.

Now that we have the current equivalent value of the amount Ashea borrowed ($F_0 = P_0$), we can easily compute her annual repayment amount over 20 years when the interest rate is fixed at 5% per year.

$$A = \$24{,}689.22(A/P, 5\%, 20) = \$24{,}689.22(0.0802) = \$1{,}980.08 \text{ per year}$$

Comment

The total principal borrowed was $4,000 + $5,000 + $6,000 + $7,000 = $22,000. Notice that a total of $17,601.60 (20 × $1,980.08 − $22,000) in interest is repaid over the entire 20-year loan period. This interest amount is close to the amount of principal originally borrowed. *Moral:* Borrow as little as possible and repay as quickly as possible to reduce interest expense! See `www.finaid.com`.

To obtain the present equivalent of a series of future cash flows subject to varying interest rates, a procedure similar to the preceding one would be utilized with a sequence of $(P/F, i_k\%, k)$ factors. In general, the present equivalent value of a cash flow occurring at the end of period N can be computed using Equation (4-31), where i_k is the interest rate for the kth period (the symbol $\prod$ means "the product of"):

$$P = \frac{F_N}{\prod_{k=1}^{N}(1 + i_k)}. \qquad (4\text{-}31)$$

For instance, if $F_4 = \$1{,}000$ and $i_1 = 10\%$, $i_2 = 12\%$, $i_3 = 13\%$, and $i_4 = 10\%$, then

$$P = \$1{,}000[(P/F, 10\%, 1)(P/F, 12\%, 1)(P/F, 13\%, 1)(P/F, 10\%, 1)]$$
$$= \$1{,}000[(0.9091)(0.8929)(0.8850)(0.9091)] = \$653.$$

4.14 Nominal and Effective Interest Rates

Very often the interest period, or time between successive compounding, is less than one year (e.g., daily, weekly, monthly, or quarterly). It has become customary to quote interest rates on an annual basis, followed by the compounding period if different from one year in length. For example, if the interest rate is 6% per interest period and the interest period is six months, it is customary to speak of this rate as "12% compounded semiannually." Here the annual rate of interest is known as the *nominal rate*, 12% in this case. A nominal interest rate is represented by r. But the actual (or effective) annual rate on the principal is not 12%, but something greater, because compounding occurs twice during the year.

Consequently, the frequency at which a nominal interest rate is compounded each year can have a pronounced effect on the dollar amount of total interest earned. For instance, consider a principal amount of $1,000 to be invested for three years at a nominal rate of 12% compounded semiannually. The interest earned during the first six months would be $\$1{,}000 \times (0.12/2) = \$60$.

Total principal and interest at the beginning of the second six-month period is

$$P + Pi = \$1{,}000 + \$60 = \$1{,}060.$$

The interest earned during the second six months would be

$$\$1{,}060 \times (0.12/2) = \$63.60.$$

Then total interest earned during the year is

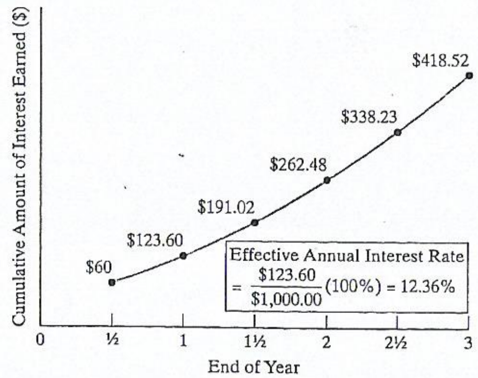
$$\$60.00 + \$63.60 = \$123.60.$$

Finally, the *effective* annual interest rate for the entire year is

$$\frac{\$123.60}{\$1{,}000} \times 100 = 12.36\%.$$

If this process is repeated for years two and three, the *accumulated* (compounded) *amount of interest* can be plotted as in Figure 4-18. Suppose that the same $1,000 had been invested at 12% compounded *monthly*, which is 1% per month.

Figure 4-18 $1,000 Compounded at a Semiannual Frequency ($r = 12\%$, $M = 2$)



The accumulated interest over three years that results from monthly compounding is shown in Figure 4-19.

The actual or exact rate of interest earned on the principal during one year is known as the *effective rate*. It should be noted that effective interest rates are always expressed on an annual basis, unless specifically stated otherwise. In this text, the effective interest rate per year is customarily designated by i and the nominal interest rate per year by r. In engineering economy studies in which compounding is annual, $i = r$. The relationship between effective interest, i, and nominal interest, r, is

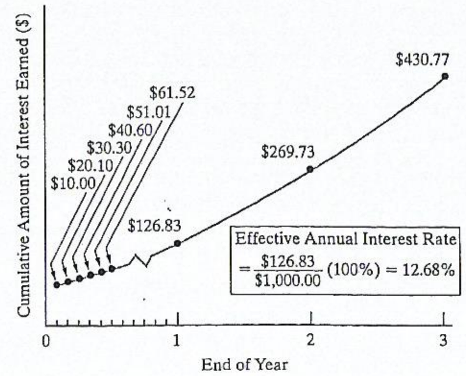
$$i = (1 + r/M)^M - 1, \quad (4\text{-}32)$$

where M is the number of compounding periods per year. It is now clear from Equation (4-32) why $i > r$ when $M > 1$.

The effective rate of interest is useful for describing the compounding effect of interest earned on interest during one year. Table 4-4 shows effective rates for various nominal rates and compounding periods.

Interestingly, the federal truth-in-lending law now requires a statement regarding the annual percentage rate (APR) being charged in contracts involving

Figure 4-19
$1,000 Compounded at a Monthly Frequency ($r = 12\%$, $M = 12$)



Effective Interest Rates for Various Nominal Rates and Compounding Frequencies

Compounding Frequency	Number of Compounding Periods per Year, M	Effective Rate (%) for Nominal Rate of 6%	8%	10%	12%	15%	24%
Annually	1	6.00	8.00	10.00	12.00	15.00	24.00
Semiannually	2	6.09	8.16	10.25	12.36	15.56	25.44
Quarterly	4	6.14	8.24	10.38	12.55	15.87	26.25
Bimonthly	6	6.15	8.27	10.43	12.62	15.97	26.53
Monthly	12	6.17	8.30	10.47	12.68	16.08	26.82
Daily	365	6.18	8.33	10.52	12.75	16.18	27.11

borrowed money. The APR is a nominal interest rate and *does not* account for compounding that may occur, or be appropriate, during a year. Before this legislation was passed by Congress in 1969, creditors had no obligation to explain how interest charges were determined or what the true cost of money on a loan was. As a result, borrowers were generally unable to compute their APR and compare different financing plans.

EXAMPLE 4-28 **Effective Annual Interest Rate**

A credit card company charges an interest rate of 1.375% per month on the unpaid balance of all accounts. The annual interest rate, they claim, is 12(1.375%) = 16.5%. What is the effective rate of interest per year being charged by the company?

Solution

Equation (4-32) is used to compute the effective rate of interest in this example:

$$i = \left(1 + \frac{0.165}{12}\right)^{12} - 1$$

$$= 0.1781, \text{ or } 17.81\%/\text{year}.$$

Note that $r = 12(1.375\%) = 16.5\%$, which is the APR. In general, it is true that $r = M(r/M)$, where r/M is the interest rate per period.

Numerous Web sites are available to assist you with personal finance decisions. Take a look at `www.dinkytown.net` and `www.bankrate.com`.

4.15 Compounding More Often than Once per Year

4.15.1 Single Amounts

If a nominal interest rate is quoted and the number of compounding periods per year and number of years are known, any problem involving future, annual, or present equivalent values can be calculated by straightforward use of Equations (4-3) and (4-32), respectively.

EXAMPLE 4-29 **Future Equivalent when Interest Is Compounded Quarterly**

Suppose that a $100 lump-sum amount is invested for 10 years at a nominal interest rate of 6% compounded quarterly. How much is it worth at the end of the 10th year?

Solution

There are four compounding periods per year, or a total of $4 \times 10 = 40$ interest periods. The interest rate per interest period is $6\%/4 = 1.5\%$. When the values are used in Equation (4-3), one finds that

$$F = P(F/P, 1.5\%, 40) = \$100.00(1.015)^{40} = \$100.00(1.814) = \$181.40.$$

Alternatively, the effective interest rate from Equation (4-32) is 6.14%. Therefore,

$$F = P(F/P, 6.14\%, 10) = \$100.00(1.0614)^{10} = \$181.40.$$

4.15.2 Uniform Series and Gradient Series

When there is more than one compounded interest period per year, the formulas and tables for uniform series and gradient series can be used *as long as* there is a cash flow at the end of each interest period, as shown in Figures 4-6 and 4-13 for a uniform annual series and a uniform gradient series, respectively.

EXAMPLE 4-30 **Computing a Monthly Auto Payment**

Stan Moneymaker has a bank loan for $10,000 to pay for his new truck. This loan is to be repaid in equal *end-of-month* installments for five years with a nominal interest rate of 12% compounded monthly. What is the amount of each payment?

Solution

The number of installment payments is 5 × 12 = 60, and the interest rate per month is 12%/12 = 1%. When these values are used in Equation (4-15), one finds that

$$A = P(A/P, 1\%, 60) = \$10{,}000(0.0222) = \$222.$$

Notice that there is a cash flow at the end of each month (interest period), including month 60, in this example.

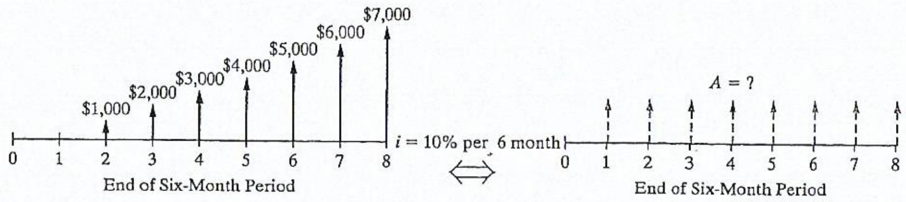
EXAMPLE 4-31 **Uniform Gradient Series and Semiannual Compounding**

Certain operating savings are expected to be 0 at the end of the first six months, to be $1,000 at the end of the second six months, and to increase by $1,000 at the end of each six-month period thereafter, for a total of four years. It is desired to find the equivalent uniform amount, A, at the end of each of the eight six-month periods if the nominal interest rate is 20% compounded semiannually.

Solution

A cash-flow diagram is given below, and the solution is

$$A = G(A/G, 10\%, 8) = \$1{,}000(3.0045) = \$3{,}004.50.$$



The symbol "⟺" in between the cash-flow diagrams indicates that the left-hand cash-flow diagram is *equivalent to* the right-hand cash-flow diagram when the correct value of A has been determined. In Example 4-31, the interest rate per six-month period is 10%, and cash flows occur every six months.

EXAMPLE 4-32 **Finding the Interest Rate on a Loan**

A loan of $15,000 requires monthly payments of $477 over a 36-month period of time. These payments include both principal and interest.

(a) What is the nominal interest rate (APR) for this loan?

(b) What is the effective interest rate per year?

(c) Determine the amount of unpaid loan principal after 20 months.

Solution

(a) We can set up an equivalence relationship to solve for the unknown interest rate since we know that $P = \$15{,}000$, $A = \$477$, and $N = 36$ months.

$$\$477 = \$15{,}000(A/P, i_{mo}, 36)$$

$$(A/P,\ i_{mo}, 36) = 0.0318$$

We can now look through Appendix C to find values of i that have an $(A/P,\ i,\ 36)$ value close to 0.0318. From Table C-3 ($i = 3/4\%$), we find $(A/P, 3/4\%, 36) = 0.0318$. Therefore,

$$i_{mo} = 0.75\% \text{ per month}$$

and

$$r = 12 \times 0.75\% = 9\% \text{ per year, compounded monthly.}$$

(b) Using Equation (4-32),

$$i_{eff} = \left(1 + \frac{0.09}{12}\right)^{12} - 1 = 0.0938 \text{ or } 9.38\% \text{ per year.}$$

(c) We can find the amount of the unpaid loan principal after 20 months by finding the equivalent value of the remaining 16 monthly payments as of month 20.

$$P_{20} = \$477(P/A, 3/4\%, 16) = \$477(15.0243) = \$7{,}166.59$$

After 20 payments have been made, almost half of the original principal amount remains. Notice that we used the monthly interest rate of 3/4% in our calculation since the cash flows are occurring monthly.

4.16 Interest Formulas for Continuous Compounding and Discrete Cash Flows

In most business transactions and economy studies, interest is compounded at the end of discrete periods, and, as has been discussed previously, cash flows are assumed to occur in discrete amounts at the end of such periods. *This practice will be used throughout the remaining chapters of this book.* However, it is evident that in most enterprises, cash is flowing in and out in an almost continuous stream. Because cash, whenever it's available, can usually be used profitably, this situation creates opportunities for very frequent compounding of the interest earned. So that this condition can be dealt with (modeled) when continuously compounded interest rates are available, the concept of continuous compounding is sometimes used in economy studies. Actually, the effect of this procedure, compared with that of discrete compounding, is rather small in most cases.

Continuous compounding assumes that cash flows occur at discrete intervals (e.g., once per year), but that compounding is continuous throughout the interval. For example, with a nominal rate of interest per year of r, if the interest is compounded M times per year, one unit of principal will amount to $[1 + (r/M)]^M$ at the end of one year. Letting $M/r = p$, we find that the foregoing expression becomes

$$\left[1+\frac{1}{p}\right]^{rp} = \left[\left(1+\frac{1}{p}\right)^p\right]^r. \tag{4-33}$$

Because

$$\lim_{p\to\infty}\left(1+\frac{1}{p}\right)^p = e^1 = 2.71828\ldots,$$

Equation (4-33) can be written as e^r. Consequently, the *continuously compounded compound amount factor (single cash flow)* at $\underline{r}\%$ nominal interest for N years is e^{rN}. Using our functional notation, we express this as

$$(F/P, \underline{r}\%, N) = e^{rN}. \tag{4-34}$$

Note that the symbol $\underline{r}$ is directly comparable to that used for discrete compounding and discrete cash flows ($i\%$), except that $\underline{r}\%$ is used to denote the nominal rate *and* the use of continuous compounding.

Since e^{rN} for continuous compounding corresponds to $(1+i)^N$ for discrete compounding, e^r is equal to $(1+i)$. Hence, we may correctly conclude that

$$i = e^r - 1. \tag{4-35}$$

By using this relationship, the corresponding values of (P/F), (F/A), and (P/A) for continuous compounding may be obtained from Equations (4-4), (4-8), and (4-10), respectively, by substituting $e^r - 1$ for i in these equations. Thus, for continuous compounding and discrete cash flows,

$$(P/F, \underline{r}\%, N) = \frac{1}{e^{rN}} = e^{-rN}; \tag{4-36}$$

$$(F/A, \underline{r}\%, N) = \frac{e^{rN}-1}{e^r-1}; \tag{4-37}$$

$$(P/A, \underline{r}\%, N) = \frac{1-e^{-rN}}{e^r-1} = \frac{e^{rN}-1}{e^{rN}(e^r-1)}. \tag{4-38}$$

Values for $(A/P, \underline{r}\%, N)$ and $(A/F, \underline{r}\%, N)$ may be derived through their inverse relationships to $(P/A, \underline{r}\%, N)$ and $(F/A, \underline{r}\%, N)$, respectively. Numerous continuous compounding, discrete cash-flow interest factors, and their uses are summarized in Table 4-5.

Because continuous compounding is used infrequently in this text, detailed values for $(A/F, \underline{r}\%, N)$ and $(A/P, \underline{r}\%, N)$ are not given in Appendix D. However, the tables in Appendix D do provide values of $(F/P, \underline{r}\%, N)$, $(P/F, \underline{r}\%, N)$, $(F/A, \underline{r}\%, N)$, and $(P/A, \underline{r}\%, N)$ for a limited number of interest rates.

Continuous Compounding and Discrete Cash Flows: Interest Factors and Symbols[a]

To Find:	Given:	Factor by which to Multiply "Given"	Factor Name	Factor Functional Symbol
For single cash flows:				
F	P	e^{rN}	Continuous compounding compound amount (single cash flow)	$(F/P, \underline{r}\%, N)$
P	F	e^{-rN}	Continuous compounding present equivalent (single cash flow)	$(P/F, \underline{r}\%, N)$
For uniform series (annuities):				
F	A	$\frac{e^{rN}-1}{e^r-1}$	Continuous compounding compound amount (uniform series)	$(F/A, \underline{r}\%, N)$
P	A	$\frac{e^{rN}-1}{e^{rN}(e^r-1)}$	Continuous compounding present equivalent (uniform series)	$(P/A, \underline{r}\%, N)$
A	F	$\frac{e^r-1}{e^{rN}-1}$	Continuous compounding sinking fund	$(A/F, \underline{r}\%, N)$
A	P	$\frac{e^{rN}(e^r-1)}{e^{rN}-1}$	Continuous compounding capital recovery	$(A/P, \underline{r}\%, N)$

[a] $\underline{r}$, nominal annual interest rate, compounded continuously; N, number of periods (years); A, annual equivalent amount (occurs at the end of each year); F, future equivalent; P, present equivalent.

Note that tables of interest and annuity factors for continuous compounding are tabulated in terms of nominal rates of interest per time period.

EXAMPLE 4-33

Continuous Compounding and Single Amounts

You have \$10,000 to invest for two years. Your bank offers 5% interest, compounded continuously for funds in a money market account. Assuming no additional deposits or withdrawals, how much money will be in that account at the end of two years?

Solution

$$F = \$10,000\ (F/P, \underline{r} = 5\%, 2) = \$10,000\ e^{(0.05)(2)} = \$10,000\ (1.1052) = \$11,052$$

Comment

If the interest rate was 5% compounded annually, the account would have been worth

$$F = \$10,000\ (F/P, 5\%, 2) = \$10,000\ (1.1025) = \$11,025.$$

EXAMPLE 4-34

Continuous Compounding and Annual Payments

Suppose that one has a present loan of \$1,000 and desires to determine what equivalent uniform EOY payments, A, could be obtained from it for 10 years if the nominal interest rate is 20% compounded continuously ($M = \infty$).

Solution

Here we utilize the formulation

$$A = P(A/P, \underline{r}\%, N).$$

Since the (A/P) factor is not tabled for continuous compounding, we substitute its inverse (P/A), which is tabled in Appendix D. Thus,

$$A = P \times \frac{1}{(P/A, \underline{20}\%, 10)} = \$1,000 \times \frac{1}{3.9054} = \$256.$$

Note that the answer to the same problem, with discrete annual compounding ($M = 1$), is

$$A = P(A/P, 20\%, 10)$$
$$= \$1,000(0.2385) = \$239.$$

EXAMPLE 4-35

Continuous Compounding and Semiannual Payments

An individual needs \$12,000 immediately as a down payment on a new home. Suppose that he can borrow this money from his insurance company. He must repay the loan in equal payments every six months over the next eight years. The nominal interest rate being charged is 7% compounded continuously. What is the amount of each payment?

Solution

The nominal interest rate per six months is 3.5%. Thus, A each six months is \$12,000$(A/P, \underline{r} = 3.5\%, 16)$. By substituting terms in Equation (4-38) and then using its inverse, we determine the value of A per six months to be \$997:

$$A = \$12,000 \left[\frac{1}{(P/A, \underline{r} = 3.5\%, 16)}\right] = \frac{\$12,000}{12.038} = \$997.$$

4.17 CASE STUDY—Understanding Economic "Equivalence"

Enrico Suarez just graduated with a B.S. in engineering and landed a new job with a starting annual salary of \$48,000. There are a number of things that he would like to do with his newfound "wealth." For starters, he needs to begin repaying his student loans (totaling \$20,000) and he'd like to reduce some outstanding balances on credit cards (totaling \$5,000). Enrico also needs to purchase a car to get to work and would like to put money aside to purchase a condo in the future. Last, but not least, he wants to put some money aside for his eventual retirement.

Our recent graduate needs to do some financial planning for which he has selected a 10-year time frame. At the end of 10 years, he'd like to have paid off his current student loan and credit card debt, as well as have accumulated \$40,000 for a down payment on a condo. If possible, Enrico would like to put aside 10% of his take home salary for retirement. He has gathered the following information to assist him in his planning.*

- Student loans are typically repaid in equal monthly installments over a period of 10 years. The interest rate on Enrico's loan is 8% compounded monthly.
- Credit cards vary greatly in the interest rate charged. Typical APR rates are close to 17%, and monthly minimum payments are usually computed using a 10-year repayment period. The interest rate on Enrico's credit card is 18% compounded monthly.
- Car loans are usually repaid over three, four, or five years. The APR on a car loan can be as low as 2.9% (if the timing is right) or as high as 12%. As a first-time car buyer, Enrico can secure a \$15,000 car loan at 9% compounded monthly to be repaid over 60 months.

* The stated problem data are current as of 2010.

- A 30-year, fixed rate mortgage is currently going for 5.75%–6.0% per year. If Enrico can save enough to make a 20% down payment on the purchase of his condo, he can avoid private mortgage insurance that can cost as much as $60 per month.
- Investment opportunities can provide variable returns. "Safe" investments can guarantee 7% per year, while "risky" investments could return 30% or more per year.
- Enrico's parents and older siblings have reminded him that his monthly take home pay will be reduced by income taxes and benefit deductions. He should not count on being able to spend more than 80% of his gross salary.

As Enrico's friend (and the one who took Engineering Economy instead of Appreciating the Art of Television Commercials), you have been asked to review his financial plans. How reasonable are his goals?

Solution

Since all repayments are done on a monthly basis, it makes sense to adopt the month as Enrico's unit of time. There are five categories for his cash flows: debt repayment, transportation costs, housing costs, other living expenses, and savings. The following paragraphs summarize his estimates of monthly expenses in each of these categories.

Debt Repayment

Enrico's student loan debt is $20,000 and is to be repaid over the next 120 months (10 years) at a nominal interest rate of 8% compounded monthly ($i_{month} = 8/12\% = 2/3\%$). His monthly loan payment is then

$$A_{Student\ Loan} = \$20{,}000(A/P, 2/3\%, 120) = \$20{,}000(0.01213) = \$242.60 \text{ per month.}$$

Enrico's credit card debt is $5,000 and is to be completely repaid over the next 120 months at a nominal interest rate of 18% compounded monthly ($i_{month} = 1.5\%$). His monthly credit card payment, assuming no additional usage, is then

$$A_{Credit\ Card} = \$5{,}000(A/P, 1.5\%, 120) = \$5{,}000(0.01802) = \$90.10 \text{ per month.}$$

Enrico's monthly debt repayment totals $242.60 + $90.10 = $332.70.

Transportation Costs

The certified pre-owned vehicle Enrico would like to buy costs $15,000. The best rate he can find as a first-time car buyer with no assets or credit history is 9% compounded monthly with a 60-month repayment period. Based on these figures, his monthly car payment will be

$$A_{Car} = \$15{,}000(A/P, 0.75\%, 60) = \$15{,}000(0.02076) = \$311.40 \text{ per month.}$$

Even though the car will be completely repaid after five years, Enrico knows that he will eventually need to replace the car. To accumulate funds for the replacement of the car, he wants to continue to set aside this amount each month for the second five years.

Insurance for this vehicle will cost $1,200 per year, and Enrico has budgeted $100 per month to cover fuel and maintenance. Thus, his monthly transportation costs total $311.40 + $1,200/12 + $100 = $511.40.

Housing Costs

A nice two-bedroom apartment near Enrico's place of work has a monthly rent of $800. The rental office provided him with a monthly utility cost estimate of $150 to cover water and electricity. Based on this information, $800 + $150 = $950 per month has been budgeted for his housing costs.

Other Living Expenses

This expense category has troubled Enrico the most. While all the previous categories are pretty straightforward, he knows that his day-to-day spending will have the most variability and that it will be tempting to overspend. Nonetheless, he has developed the following estimates of monthly living expenses not already covered in the other categories:

Food	$200
Phone	$70
Entertainment	$100
Miscellaneous (clothing, household items)	$150
Subtotal	$520

Savings

Enrico wants to accumulate $40,000 over the next 10 years to be used as a down payment on a condo. He feels that if he chooses relatively "safe" investments (CDs and bonds), he should be able to earn 6% compounded monthly on his savings. He must then set aside

$$A_{Condo} = \$40{,}000(A/F, 0.5\%, 120) = \$40{,}000(0.00610) = \$244.00 \text{ per month}$$

to reach his goal.

Enrico's gross monthly starting salary is $48,000/12 = $4,000. Based on the information gathered from his family, he estimates his monthly net (take home) pay to be $4,000(0.80) = $3,200. His monthly retirement savings will then be $3,200(0.10) = $320. Thus, the total amount to be set aside each month for the future is $244 + $320 = $564.

Monthly Financial Plan

Based on the preceding calculations, the following table summarizes Enrico's monthly financial plan.

	Net Income	Expense
Salary	$3,200	
Debt repayment		$332.70
Transportation costs		511.40
Housing costs		950.00
Living expenses		520.00
Savings		564.00
Total	$3,200	$2,878.10

Enrico is aware that he has not explicitly accounted for price increases over the next 10 years; however, neither has he increased his annual salary. He is hopeful that, if he works hard, his annual salary increases will at least cover the increase in the cost of living.

You congratulate your friend on his efforts. You feel that he has covered the major areas of expenses, as well as savings for the future. You note that he has $3,200 − $2,878.10 = $321.90 of "extra" money each month to cover unanticipated costs. With this much leeway, you feel that he should have no problem meeting his financial goals.

4.18 Summary

Chapter 4 presented the fundamental time value of money relationships that are used throughout the remainder of this book. Considerable emphasis has been placed on the concept of equivalence—if two cash flows (or series of cash flows) are equivalent for a stated interest rate, you are willing to trade one for the other. Formulas relating present and future amounts to their equivalent uniform, arithmetic, and geometric series have been presented. You should feel comfortable with the material in this chapter before embarking on your journey through subsequent chapters. Important abbreviations and notation are listed in Appendix B.

In the next chapter, we will see how to apply time value of money relationships to the estimated cash flows of a project (the topic of Chapter 3) to arrive at a measure of its economic worth.

Problems

The number in parentheses that follows each problem refers to the section from which the problem is taken.

4-1. If $100 is placed in an account that earns 8% compounded quarterly, what will its worth be in 8 years? **(4.3)**

4-2. You borrow $200 from a family member and agree to pay it back in six months. Because you are part of the family, you are only being charged simple interest at the rate of 1% per month. How much will you owe after six months? How much of this is interest? **(4.2)**

4-3. $2,000 is deposited in a savings account that pays 10% compounded annually. How much will the account be worth in 15 years? **(4.3)**

4-4. The average price of gold in 2010 is $400 per ten grams, while in 2000 it was $300. What was the average annual rate of increase in the price of gold over the 10-year period? **(4.5)**

4-5. A father, whose dream is to see his son grow up to be an engineer, plans to invest a certain sum towards his son's education upon his birth. He will need to withdraw $12,000 each year from the 21st to the 24th year of his son's life. How much should he invest, if the rate of interest is 10% compounded annually? **(4.4)**

4-6. Suppose that, in Plan 1 of Table 4-1, $8,500 of the original unpaid balance is to be repaid at the end of months two and four only. How much total interest would have been paid by the end of month four? **(4.4)**

4-7. Refer to Plan 2 in Table 4-1. This is the customary way to pay off loans on automobiles, house mortgages, etc. A friend of yours has financed $24,000 on the purchase of a new automobile, and the annual interest rate is 12% (1% per month). **(4.4)**

a. Monthly payments over a 60-month loan period will be how much?

b. How much interest and principal will be paid in the third month of this loan?

4-8. Rick wants to have earned $10,000 at the end of 6 years. How much must he deposit today in order to do so, if his rate of interest is fixed at 8% compounded annually? **(4.4)**

4-9. Peter needs to finance a business expansion program. One bank offers to lend the required $1 million at a quoted annual rate of 10% compounded quarterly. A second lender offers 9% compounded daily. Assume that Peter will pay both interest and principal at the end of one year. What is the difference in the effective annual rates charged by the two banks? **(4.28)**

4-10. A lump-sum loan of $5,000 is needed by Chandra to pay for college expenses. She has obtained small consumer loans with 12% interest per year in the past to help pay for college. But her father has advised Chandra to apply for a PLUS student loan charging only 8.5% interest per year. If the loan will be repaid in full in five years, what is the difference in total interest accumulated by these two types of student loans? **(4.6)**

4-11. You invest $25,000 in a stock-based mutual fund. This fund should earn (on average) 10% per year over a long period of time. How much should your investment be worth in 25 years? **(4.6)**

4-12. What is the present equivalent of $18,000 to be received in 15 years when the interest rate is 7% per year? **(4.6)**

4-13. A student's average tuition fee is $1,000 per year over 20 years of student life. Find the total fees the student would pay at the end of 20 years, if the interest rate is 5% compounded annually. **(4.7)**

4-14. Mr. PBS is looking forward to retiring in 2010. He wants to invest some money so that he can earn $10,000 per year for the next 20 years of his life. If the rate of interest is 8% compounded annually, what is the amount of the current investment? **(4.4)**

4-15. Mr. Smith has saved $1,200 each year for 20 years. A year after the saving period ended, Mr. Smith withdrew $7,500 each year for a period of five years. In the sixth and seventh years, he only withdrew $4,500 per year. In the eighth year, he decided to withdraw the remaining money in his account. If the interest rate was 8% per year throughout the whole period, what was the amount he withdrew at the end of the eighth year? **(4.10)**

4-16. Use the rule of 72 to determine how long it takes to accumulate $10,000 in a savings account when $P =$ $5,000 and $i = 10\%$ per year. **(4.6)**

> Rule of 72: The time (years) required to double the value of a lump-sum investment that is allowed to compound is approximately
>
> 72 ÷ annual interest rate (as a %).

4-17. Suppose you are considering borrowing $140,000 to finance your house. The annual rate of interest is 15% and payments are made monthly. If the mortgage is on a 30-year schedule, what are the monthly payments you would have to make? **(4.15)**

4-18. Adam borrowed $50,000 from his credit union to purchase a house. The interest rate on a home loan is 1% per month and he anticipates making a total of 240 monthly payments. How much does he pay every month? **(4.11)**

4-19. An enterprising student invests $1,000 at an annual interest rate that will grow the original investment to $2,000 in four years. In four more years, the amount will grow to $4,000, and this pattern of doubling every four years repeats over a total time span of 36 years. How much money will the student *gain* in 36 years? What is the magical annual interest rate the student is earning? (4.6)

4-20. You are 20 years old and planning for retirement, when you will need $15,000 per year after retirement up to the age of 80. You plan to make annual deposits into your account till you turn 60. If the interest rate is a constant 12% during the 40 year period, how much must you deposit annually into your account? (4.7)

4-21. A good stock-based mutual fund should earn at least 10% per year over a long period of time. Consider the case of Barney and Lynn, who were overheard gloating (for all to hear) about how well they had done with their mutual fund investment. "We turned a $25,000 investment of money in 1982 into $100,000 in 2007." (4.6)

a. What return (interest rate) did they really earn on their investment? Should they have been bragging about how investment-savvy they were?

b. Instead, if $1,000 had been invested each year for 25 years to accumulate $100,000, what return did Barney and Lynn earn?

4-22. In 1972, the maximum earnings of a worker subject to social security tax was $9,000. The maximum earnings subject to social security tax in 2008 is $102,000. What compounded annual increase has been experienced over this 36-year period of time? How does it compare with a 3% annual increase in the consumer price index (CPI) over this same time interval? (4.6)

4-23. At a certain state-supported university, annual tuition and fees have risen dramatically in recent years as shown in the table below. (4.6)

Year	Tuition and Fees	Consumer Price Index
1982–1983	$827	96.5
1987–1988	$1,404	113.6
1993–1994	$2,018	144.5
2003–2004	$4,450	184.0
2005–2006	$5,290	198.1 (est.)

Source: www.bls.gov.

a. If all tuition and fees are paid at the beginning of each academic year, what is the compound annual rate of increase from 1982 to 2005?

b. What is the annual rate of increase from 1993 to 2005?

c. How do the increases in Parts (a) and (b) compare with the CPI for the same period of time?

4-24. Mr. Greg wants to purchase his dream house, which will cost $600,000 in 15 years. He is going to put aside $40,000 each year for 10 years towards this. At what interest rate must he invest his money to achieve his goal of having enough to purchase the house after 10 years? (4.7)

4-25. Suppose you make 15 equal annual deposits of $1,000 each into a bank account paying 5% interest per year. The first deposit will be made one year from today. How much money can be withdrawn from this bank account immediately after the 15th deposit? (4.7)

4-26. James plans to retire in 30 years. He has $15,000 to invest and also plans to begin depositing $500 each month at the end of this month. If he continues to invest $500 each month until he retires how much will he have accumulated by the time he retires? Assume he can earn interest at an annual rate of 8% compounded monthly. (4.7)

4-27. A credit card company wants your business. If you accept their offer and use their card, they will deposit 1% of your monetary transactions into a savings account that will earn a guaranteed 5% per year. If your annual transactions total an average of $20,000, how much will you have in this savings plan after 15 years? (4.7)

4-28. One of life's great lessons is to start early and save all the money you can! If you save two dollars today and two dollars each and every day thereafter until you are 60 years old, how much money will you accumulate (say, $730 per year for 35 years) if the annual interest rate is 7%? (4.7)

4-29. Liam O'Kelly is 20 years old and is thinking about buying a term life insurance policy with his wife as the beneficiary. The quoted annual premium for Liam is $8.48 per thousand dollars of insurance coverage. Because Liam wants a $100,000 policy (which is 2.5 times his annual salary), the annual premium would be $848, with the first payment due immediately (i.e., at age 21).

A friend of Liam's suggests that the $848 annual premium should be deposited in a good mutual fund rather than in the insurance policy. "If the mutual fund earns 10% per year, you can become a millionaire by the time you retire at age 65," the friend advises. (4.7)

a. Is the friend's statement really true?

b. Discuss the trade-off that Liam is making if he decides to invest his money in a mutual fund.

4-30. Luis wants to have $2,000,000 in net worth when he retires. To achieve this goal, he plans to invest $10,000 each year (starting one year from now) into an account that earns 10% interest compounded annually. The amount of time before Luis can retire as a multimillionaire is how many years? (4.7)

4-31. Twelve payments of $10,000 each are to be repaid monthly at the end of each month. The monthly interest rate is 2%. (4.7)

a. What is the present equivalent (i.e., P_0) of these payments?

b. Repeat Part (a) when the payments are made at the beginning of the month. Note that the present equivalent will be at the same time as the first monthly payment.

c. Explain why the present equivalent amounts in Parts (a) and (b) are different.

4-32. You are saving up for your ten-year-old daughter's wedding (when she turns 21) and you want to invest $10,000 per year for five years from now, at the rate of interest of 15% per year. How much will you have at the time of the wedding? (4.7)

4-33. Automobiles of the future will most likely be manufactured largely with carbon fibers made from recycled plastics, wood pulp, and cellulose. Replacing half the ferrous metals in current automobiles could reduce a vehicle's weight by 60% and fuel consumption by 30%. One impediment to using carbon fibers in cars is *cost*. If the justification for the extra sticker price of carbon-fiber cars is solely based on fuel savings, how much extra sticker price can be justified over a six-year life span if the carbon-fiber car would average 39 miles per gallon of gasoline compared to a conventional car averaging 30 miles per gallon? Assume that gasoline costs $3.00 per gallon, the interest rate is 20% per year, and 117,000 miles are driven uniformly over six years. (4.7)

4-34. It is estimated that a certain piece of equipment can save $22,000 per year in labor and materials costs. The equipment has an expected life of five years and no market value. If the company must earn a 15% annual return on such investments, how much could be justified now for the purchase of this piece of equipment? Draw a cash-flow diagram from the company's viewpoint. (4.7)

4-35. Spivey just won the Powerball lottery! The $20,000,000 jackpot will be paid in 20 annual installments of $1,000,000 each, with the first payment to be paid immediately. Spivey's opportunity cost of capital (interest rate) is 6% per year. What is the present equivalent of Spivey's lottery winnings at the time of the first payment? (4.7)

4-36. John Smith took out a student loan to complete his four-year engineering degree. He borrowed $5000, four years ago when the interest rate was 5% per year. A further $6000 was borrowed 3 years ago at 3% per year. Two years ago he borrowed $6000 at 6% and last year $7000 was borrowed at 8% per year. If Smith makes annual payments to repay his total debt payments up to 10 years with 8% fixed annual interest rate. What is the amount of each payment? (4.6)

4-37. A large retailer is going to run an experiment at one of its stores to see how cost effective it will be to imbed merchandise with small radio frequency identification (RFID) chips. Shoppers will select items from the store and walk out without stopping to pay at a checkout lane. The RFID chips will record what items are taken and will automatically deduct their price from shoppers' bank accounts. The retailer expects to save the cost of staffing 12 checkout lanes, which amounts to $100,000 per year. How much can the retailer afford to spend on its RFID investment if the system has a life of 10 years and no residual value? The retailer's interest rate is 15% per year. (4.7)

4-38. The U.S. stock market has returned an average of about 9% per year since 1900. This return works out to a real return (i.e., adjusted for inflation) of approximately 6% per year. (4.7)

a. If you invest $100,000 and you earn 6% a year on it, how much real purchasing power will you have in 30 years?

b. If you invest $5,000 per year for 20 years, how much real purchasing power will you have at the end of 30 years? The interest rate is 6% per year.

4-39. A series of 10 annual payments of $8,000 is equivalent to four equal payments at the end of years 6, 10, 12, and 15 when the interest rate is 10% compounded annually. What is the amount of these four payments? (4.11)

4-40. Qwest Airlines has implemented a program to recycle all plastic drink cups used on their aircraft. Their goal is to generate $5 million by the end of the recycle program's five-year life. Each recycled cup can be sold for $0.005 (1/2 cent). (4.7)

a. How many cups must be recycled annually to meet this goal? Assume uniform annual plastic cup usage and a 0% interest rate.

b. Repeat Part (a) when the annual interest rate is 15%.

c. Why is the answer to Part (b) less than the answer to Part (a)?

4-41. Show that the following relationship is true: $(A/P, i\%, N) = i/[1 - (P/F, i\%, N)]$. **(4.7)**

4-42. A bank is designing a new account that pays interest quarterly. They wish to pay, effectively, 16% per year on this account. They want to advertise the annual percentage rate on this new account instead of the effective rate since its competitors state their interest on annualized basis. What is the APR that corresponds to an effective rate of 16% for this new account? **(4.14)**

4-43. A large automobile manufacturer has developed a continuous variable transmission (CVT) that provides smooth shifting and enhances fuel efficiency by 2 mpg of gasoline. The extra cost of a CVT is $800 on the sticker price of a new car. For a particular model averaging 28 miles per gallon with the CVT, what is the cost of gasoline (dollars per gallon) that makes this option affordable when the buyer's interest rate is 10% per year? The car will be driven 100,000 miles uniformly over an eight-year period. **(4.7)**

4-44. What present amount, at 12% interest compounded annually, is equivalent to this series of payments: Year 1—$10,000, Year 2—$9,000, Year 3—$8,000, Year 4—$7,000, Year 5—$6,000? Solve using the gradient factors and then using only single payment present worth factors. **(4.22)**

4-45. A loan company is offering business loans of $10,000 under a scheme that requires borrowers to pay off the loan in 36 monthly instalments of $500 each. What is the effective annual rate of interest this loan company is charging its customers? **(4.17, 4.28)**

4-46. $15,000 was invested, from which can be withdrawn a geometric gradient series of annual payments decreasing at the rate of 10% per year. The first payment received was $3,000 and it occurred after the investment followed by the remaining five payments (total of six withdrawals). What is the rate of interest earned on this investment? **(4.12)**

4-47. The Golden Gate Bridge in San Francisco was financed with construction bonds sold for $35 million in 1931. These were 40-year bonds, and the $35 million principal plus almost $39 million in interest were repaid in total in 1971. If interest was repaid as a lump sum, what interest rate was paid on the construction bonds? **(4.6)**

4-48. What single amount at the end of the fifth year is equivalent to a uniform annual series of $10,000 per year for 12 years? The interest rate is 7% compounded annually. **(4.11)**

4-49. Consider the accompanying cash-flow diagram. (See Figure P4-49.) **(4.7)**

a. If $P = \$1{,}000$, $A = \$200$, and $i\% = 12\%$ per year, then $N = ?$

b. If $P = \$1{,}000$, $A = \$200$, and $N = 10$ years, then $i = ?$

c. If $A = \$200$, $i\% = 12\%$ per year, and $N = 5$ years, then $P = ?$

d. If $P = \$1{,}000$, $i\% = 12\%$ per year, and $N = 5$ years, then $A = ?$

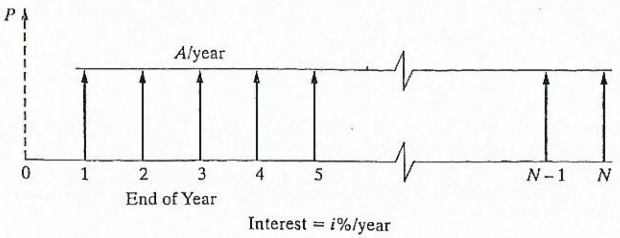


Figure P4-49 Figure for Problem 4-49

4-50. Suppose that your rich uncle has $1,000,000 that he wishes to distribute to his heirs at the rate of $100,000 per year. If the $1,000,000 is deposited in a bank account that earns 6% interest per year, how many years will it take to completely deplete the account? How long will it take if the account earns 8% interest per year instead of 6%? **(4.7)**

4-51. A cash flow series increases geometrically at the rate of 6% per year. The initial payment in the first year is $5000 with increasing annual payment ending at the end of 20 years. The interest rate in effect is 15% compounded annually for the first eight years and 5% compounded annually for the 12 remaining years. Find the present amount that is equivalent to the cash flow? **(4.12)**

4-52. An annually increasing uniform gradient series begins at the end of the 1st year with a payment of $1500 which increases each year by G and ends after the 20th year. What is the value of gradient G that makes the gradient series equivalent to an equal series of payment of $20,000 each year for 20 years at 10% compounded annually? **(4.12)**

4-53. Sam and Smith want to buy a car that costs $100,000. The bank will lend them the money at 10% APR for 30 years, with monthly payments to begin in one month. How much will their monthly payments be? **(4.28)**

4-54. Kris borrows money in her senior year to buy a new car. The car dealership allows her to defer payments for 12 months, and Kris makes 36 end-of-month payments thereafter. If the original note (loan) is for $24,000 and interest is 1/2% per month on the unpaid balance, how much will Kris' payments be? **(4.9)**

4-55. A saving system yields $5,000 at the end of the first six months, and a further $1,000 increase every six months for a total of 5 years. Find the equivalent amount A, at the end of each six month period, if the nominal interest rate is 20% compounded semi-annually? **(4.21)**

4-56. You are currently investing your money in a bank account which has a nominal annual rate of 8% compounded annually. If you invest $2,000 today, how many years will it take for your account to grow to $10,000? **(4.9)**

4-57. What lump sum of money must be deposited into a bank account at the present time so that $500 per month can be withdrawn for five years, with the first withdrawal scheduled for six years from today? The interest rate is 3/4% per month. (*Hint:* Monthly withdrawals begin at the end of the month 72.) **(4.9)**

4-58. You can buy a machine for $100,000 that will produce a net income, after operating expenses, of $10,000 per year. If you plan to keep the machine for four years, what must the market (resale) value be at the end of four years to justify the investment? You must make a 15% annual return on your investment. **(4.7)**

4-59. Major overhaul expenses of $5,000 each are anticipated for a large piece of earthmoving equipment. The expenses will occur at EOY four and will continue every three years thereafter up to and including year 13. The interest rate is 12% per year. **(4.10)**

a. Draw a cash-flow diagram.

b. What is the present equivalent of the overhaul expenses at time 0?

c. What is the annual equivalent expense during only years 5–13?

4-60. Maintenance costs for a small bridge with an expected 50-year life are estimated to be $1,000 each year for the first 5 years, followed by a $10,000 expenditure in the year 15 and a $10,000 expenditure in the year 30. If $i = 10\%$ per year, what is the equivalent uniform annual cost over the entire 50-year period? **(4.10)**

4-61. It is estimated that you will pay about $80,000 into the social security system (FICA) over your 40-year work span. For simplicity, assume this is an annuity of $2,000 per year, starting with your 26th birthday and continuing through your 65th birthday. **(4.10)**

a. What is the future equivalent worth of your social security savings when you retire at age 65 if the government's interest rate is 6% per year?

b. What annual withdrawal can you make if you expect to live 20 years in retirement? Let $i = 6\%$ per year.

4-62. A father wants to set up a bank account that will pay his daughter $15,000 at the end-of-quarter (EOQ) 4 and $25,000 at EOQ 8. He will fund this account by making quarterly payments of $X from the present (time zero) through EOQ 7. **(4.10)**

a. Draw a cash-flow diagram from the father's viewpoint.

b. If the quarterly percentage rate is 2%, what is the value of $X that must be deposited into the account?

4-63. The Turners have 10 years to save a lump-sum amount for their child's college education. Today a four-year college education costs \$75,000, and this is expected to increase by 10% per year into the foreseeable future. (4.10)

a. If the Turners can earn 6% per year on a conservative investment in a highly rated tax-free municipal bond, how much money must they save each year for the next 10 years to afford to send their child to college?

b. If a certain college will "freeze" the cost of education in 10 years for a lump-sum of current value \$150,000, is this a good deal?

4-64. Baby boomers can save up to \$22,000 per year in a 401(k) account. If Eileen's starting balance at age 50 is \$200,000 and she saves the full amount available to her, how much money will she have saved when she is 65 years old (after 15 years of saving)? The interest rate is 7% per year. (4.10)

4-65. An auto dealership is running a promotional deal whereby they will replace your tires free of charge for the life of the vehicle when you purchase your car from them. You expect the original tires to last for 30,000 miles, and then they will need replacement every 30,000 miles thereafter. Your driving mileage averages 15,000 miles per year. A set of new tires costs \$400. If you trade in the car at 150,000 miles with new tires then, what is the lump-sum present value of this deal if your personal interest rate is 12% per year? (4.10)

4-66. Transform the cash flows on the left-hand side of the accompanying diagram (see Figure P4-66) to their equivalent amount, F, shown on the right-hand side. The interest rate is 8% per year. (4.10)

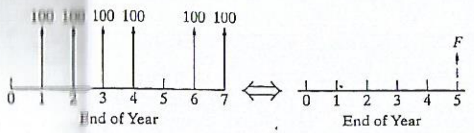


Figure P4-66 Figure for Problem 4-66

4-67. In recent years, the United States has gone from being a "positive savings" nation to a "negative savings" nation (i.e., we spend more money than we earn). Suppose a typical American household spends \$10,000 more than it makes and it does this for eight consecutive years. If this debt will be financed at an interest rate of 15% per year, what annual repayment will be required to repay the debt over a 10-year period (repayments will start at EOY 9)? (4.10)

4-68. Determine the value of W on the right-hand side of the accompanying diagram (see Figure P4-68) that makes the two cash-flow diagrams equivalent when $i = 12\%$ per year. (4.10)

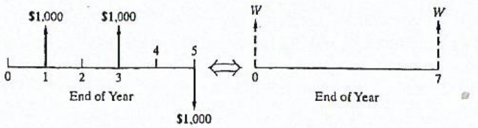


Figure P4-68 Figure for Problem 4-68

4-69. A friend of yours just bought a new sports car with a \$5,000 down payment, and her \$30,000 car loan is financed at an interest rate of 0.75% per month for 48 months. After 2 years, the "Blue Book" value of her vehicle in the used-car marketplace is \$15,000. (4.10)

a. How much does your friend still owe on the car loan immediately after she makes her 24th payment?

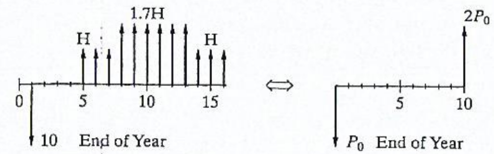
b. Compare your answer to Part (a) to \$15,000. This situation is called being "upside down." What can she do about it? Discuss your idea(s) with your instructor.

4-70. A certain fluidized-bed combustion vessel has an investment cost of \$100,000, a life of 10 years, and negligible market (resale) value. Annual costs of materials, maintenance, and electric power for the vessel are expected to total \$10,000. A major relining of the combustion vessel will occur during the fifth year at a cost of \$30,000. If the interest rate is 15% per year, what is the lump-sum equivalent cost of this project at the present time? (4.10)

4-71. It costs \$30,000 to retrofit the gasoline pumps at a certain filling station so the pumps can dispense E85 fuel (85% ethanol and 15% gasoline). If the station makes a profit of \$0.08 per gallon from selling E85 and sells an average of 20,000 gallons of E85 per month, how many months will it take for the owner to recoup her \$30,000 investment in the retrofitted pumps? The interest rate is 1% per month. (4.10)

4-72. An expenditure of \$20,000 is made to modify a material-handling system in a small job shop. This modification will result in first-year savings of \$2,000, a second-year savings of \$4,000, and a savings of \$5,000 per year thereafter. How many years must the system last if an 18% return on investment is required? The system is tailor made for this job shop and has no market (salvage) value at any time. (4.10)

4-73. Determine the value of P_0, as a function of H, for these two investment alternatives to be equivalent at an interest rate of $i = 15\%$ per year: (4.10)



4-74. Find the uniform annual amount that is equivalent to a uniform gradient series in which the first year's payment is \$500, the second year's payment is \$600, the third year's payment is \$700, and so on, and there is a total of 20 payments. The annual interest rate is 8%. (4.11)

4-75. Suppose that annual income from a rental property is expected to start at \$1,300 per year and decrease at a uniform amount of \$50 each year after the first year for the 15-year expected life of the property. The investment cost is \$8,000, and i is 9% per year. Is this a good investment? Assume that the investment occurs at time zero (now) and that the annual income is first received at EOY one. (4.11)

4-76. For a repayment schedule that starts at EOY four at \$Z and proceeds for years 4 through 10 at \$2Z, \$3Z, ..., what is the value of Z if the principal of this loan is \$10,000 and the interest rate is 7% per year? Use a uniform gradient amount (G) in your solution. (4.11)

4-77. Refer to the accompanying cash-flow diagram (see Figure P4-77), and solve for the unknown quantity in Parts (a) through (d) that makes the equivalent value of cash outflows equal to the equivalent value of the cash inflow, F. (4.11)

a. If $F = \$10{,}000$, $G = \$600$, and $N = 6$, then $i = ?$

b. If $F = \$10{,}000$, $G = \$600$, and $i = 5\%$ per period, then $N = ?$

c. If $G = \$1{,}000$, $N = 12$, and $i = 10\%$ per period, then $F = ?$

d. If $F = \$8{,}000$, $N = 6$, and $i = 10\%$ per period, then $G = ?$

4-78. You owe your best friend \$2,000. Because you are short on cash, you offer to repay the loan over 12 months under the following condition. The first payment will be \$100 at the end of month one. The second payment will be \$100 + G at the end of month two. At the end of month three, you'll repay \$100 + 2G. This pattern of increasing G amounts will continue for all remaining months. (4.11)

a. What is the value of G if the interest rate is 0.5% per month?

b. What is the equivalent uniform monthly payment?

c. Repeat Part (a) when the first payment is \$150 (i.e., determine G).

4-79. In the accompanying diagram, Figure P4-79 (p. 192), what is the value of K on the left-hand cash-flow diagram that is equivalent to the right-hand cash-flow diagram? Let $i = 12\%$ per year. (4.11)

4-80. Calculate the future equivalent at the end of 2012, at 8% per year, of the following series of cash flows in Figure P4-80 (p. 192): [Use a uniform gradient amount (G) in your solution.] (4.11)

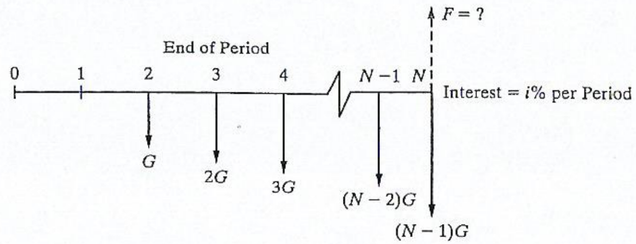


Figure P4-77 Figure for Problem 4-77

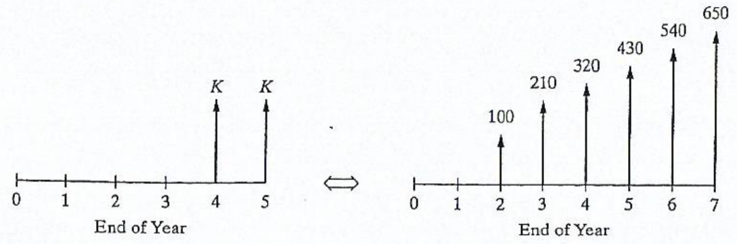


Figure P4-79 Figure for Problem 4-79

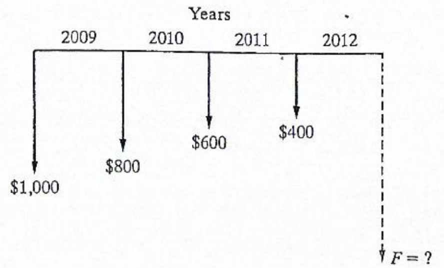


Figure P4-80 Figure for Problem 4-80

4-81. Suppose that the parents of a young child decide to make annual deposits into a savings account, with the first deposit being made on the child's fifth birthday and the last deposit being made on the 15th birthday. Then, starting on the child's 18th birthday, the withdrawals as shown will be made. If the effective annual interest rate is 8% during this period of time, what are the annual deposits in years 5 through 15? Use a uniform gradient amount (G) in your solution. (See Figure P4-81.) **(4.11)**

4-82. The heat loss through the exterior walls of a certain poultry processing plant is estimated to cost the owner $3,000 next year. A salesman from Superfiber Insulation, Inc., has told you, the plant engineer, that he can reduce the heat loss by 80% with the installation of $18,000 worth of Superfiber now. If the cost of heat loss rises by $200 per year (uniform gradient) after the next year and the owner plans to keep the present building for 15 more years, what would you recommend if the interest rate is 10% per year? **(4.11)**

4-83. What value of N comes closest to making the left-hand cash-flow diagram of the accompanying figure, Figure P4-83 (p. 193), equivalent to the one on the right? Let $i = 15\%$ per year. Use a uniform gradient amount (G) in your solution. **(4.11)**

4-84. A retail outlet is being designed in a strip mall in Nebraska. For this outlet, the installed fiberglass insulation to protect against heat loss in the winter and heat gain in the summer will cost an estimated $100,000. The annual savings in energy due to the insulation will be $18,000 at EOY one in the 10-year life of the outlet, and these savings will increase by 12% each year thereafter. If the annual interest rate is 15%, is the cost of the proposed amount of insulation justified? **(4.12)**

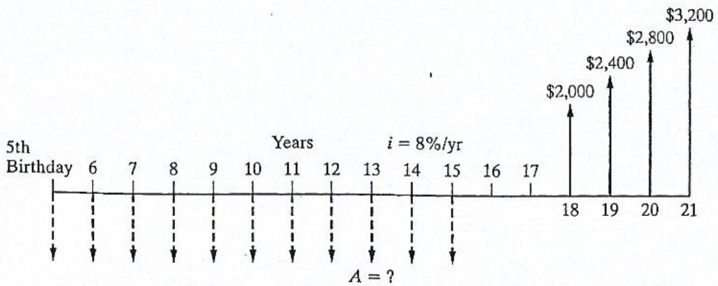


Figure P4-81 Figure for Problem 4-81

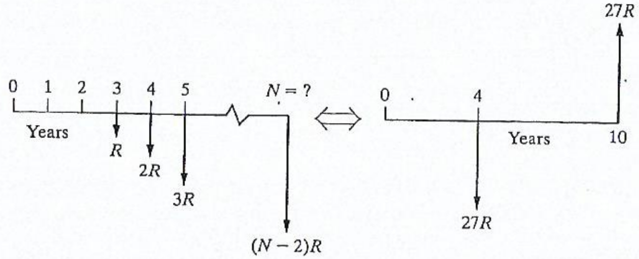
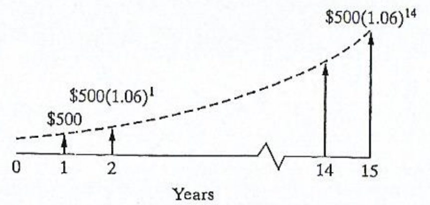


Figure P4-83 Figure for Problem 4-83

4-85. You are the manager of a large crude-oil refinery. As part of the refining process, a certain heat exchanger (operated at high temperatures and with abrasive material flowing through it) must be replaced every year. The replacement and downtime cost in the first year is $175,000. This cost is expected to increase due to inflation at a rate of 8% per year for five years, at which time this particular heat exchanger will no longer be needed. If the company's cost of capital is 18% per year, how much could you afford to spend for a higher quality heat exchanger so that these annual replacement and downtime costs could be eliminated? **(4.12)**

4-86. A geometric gradient has a positive cash flow of $1,000 at EOY zero (now), and it increases 5% per year for the following five years. Another geometric gradient has a positive value of $2,000 at EOY one, and it decreases 6% per year for years two through five. If the annual interest rate is 10%, which geometric gradient would you prefer? **(4.12)**

4-87. A geometric gradient that increases at $\bar{f} = 6\%$ per year for 15 years is shown in the accompanying diagram. The annual interest rate is 12%. What is the present equivalent value of this gradient? **(4.12)**



4-88. A small company heats its building and spends $8,000 per year on natural gas for this purpose. Cost increases of natural gas are expected to be 10% per year starting one year from now (i.e., the first cash flow is $8,800 at EOY one). Their maintenance on the gas furnace is $345 per year, and this expense is expected to increase by 15% per year starting one year from now. If the planning horizon is 15 years, what is the total annual equivalent expense for operating and maintaining the furnace? The interest rate is 18% per year. **(4.12)**

4-89. In a geometric sequence of annual cash flows starting at EOY *zero*, the value of A_0 is $1,304.35 (which *is* a cash flow). The value of the last term in the series, A_{10}, is $5,276.82. What is the equivalent value of A for years 1 through 10? Let $i = 20\%$ per year. **(4.12)**

4-90. Amy Parker, a 22-year-old and newly hired marine biologist, is quick to admit that she does not plan to keep close tabs on how her 401(k) retirement plan will grow with time. This sort of thing does not really interest her. Amy's contribution, plus that of her employer, amounts to $2,200 per year starting at age 23. Amy expects this amount to increase by 3% each year until she retires at the age of 62 (there will be 40 EOY payments). What is the compounded future value of Amy's 401(k) plan if it earns 7% per year? **(4.12)**

4-91. An electronic device is available that will reduce this year's labor costs by $10,000. The equipment is expected to last for eight years. If labor costs increase at an average rate of 7% per year and the interest rate is 12% per year. **(4.12)**

a. What is the maximum amount that we could justify spending for the device?

b. What is the uniform annual equivalent value (A) of labor costs over the eight-year period?

4-92. An amou... must be invested now to allow withdrawals of $... per year for the next 15 years and to permit $300 t... withdrawn starting at the end of year 6 and contin... over the remainder of the 15-year period as the $3... increases by 6% per year thereafter. That is, the wit... wal at EOY seven will be $318, $337.08 at EOY ... t, and so forth for the remaining years. The inter... ate is 12% per year. *Hint:* Draw a cash-flow diagra... (4.12)

4-93. Consider a... EOY geometric gradient, which lasts for eight years, ... hose initial value at EOY one is $5,000 and $\bar{f}$ ... % per year thereafter. Find the equivalent unif... gradient amount over the same period if the ini... value of the cash flows at the end of year one is $4,... Complete the following questions to determine th... ue of the gradient amount, G. The interest rate is 8... r year. (4.12)

a. What is P_0 f... e geometric gradient series?

b. What is P_0' ... e uniform (arithmetic) gradient of cash flows?

c. What is the ... e of G?

4-94. You rece... twenty year annuity which pays $50 annually. T... rst payment will be received at the end of the first ... You decide to invest each payment in an account ... ys 10% compounded continuously. What will the v... of your account be at the end of the thirtieth year? (...)

4-95. A person ... made an arrangement to borrow $1,000 now and ... other $1,000 two years hence. The entire obligatio... o be repaid at the end of four years. If the projected ... erest rates in years one, two, three, and four are 10... %, 12%, and 14%, respectively, how much will be re... d as a lump-sum amount at the end of four years? (...)

4-96. Suppose that you have a money market certificate earning an annual rate of interest, which varies over time as follows:

Year k	1	2	3	4	5
i_k	6%	4%	2%	2%	5%

If you invest $10,000 in this certificate at the beginning of year one and do not add or withdraw any money for five years, what is the value of the certificate at the end of the fifth year? (4.13)

4-97. Determine the present equivalent value of the cash-flow diagram of Figure P4-97 when the annual interest rate, i_k, varies as indicated. (4.13)

4-98. Mary's credit card situation is out of control because she cannot afford to make her monthly payments. She has three credit cards with the following loan balances and APRs: Card 1, $4,500, 21%; Card 2, $5,700, 24%; and Card 3, $3,200, 18%. Interest compounds monthly on all loan balances. A credit card loan consolidation company has captured Mary's attention by stating they can save Mary 25% per month on her credit card payments. This company charges 16.5% APR. Is the company's claim correct? (4.14)

4-99. Compute the effective annual interest rate in each of these situations: (4.14)

a. 10% nominal interest, compounded semiannually.

b. 10% nominal interest compounded quarterly.

c. 10% nominal interest compounded weekly.

4-100. What equal amount must be deposited into a sinking fund to accumulate $60,000 in 20 years if the rate of interest is 15% compounded continuously? (4.33, 4.34)

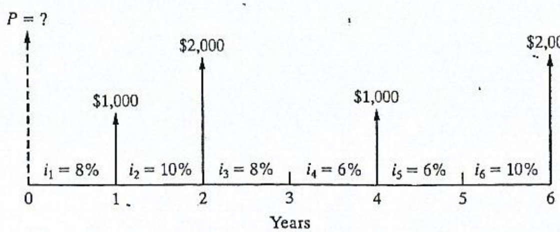


Figure P4-97 Figure for Problem 4-97

4-101. A large bank has increased its annual percentage rate (APR) on credit cards to 30%. This move was necessary because of the "additional risks" faced by the bank in a weak economy. If monthly compounding is in effect, what is the effective annual interest rate being charged by the bank? (4.15)

4-102. A mortgage banking company has been evaluating the merits of a 50-year mortgage (in addition to their popular 30-year mortgage). The basic idea is to reduce the monthly payment and make home ownership more affordable. The APR of either mortgage is 6% and the compounding is monthly. (4.15)

a. For a mortgage loan of $300,000, what is the difference in monthly payment for the 30-year mortgage and the 50-year mortgage?

b. What is the difference in total interest paid between the two mortgages?

4-103. Determine the current amount of money that must be invested at 12% nominal interest, compounded monthly, to provide an annuity of $10,000 (per year) for 6 years, starting 12 years from now. The interest rate remains constant over this entire period of time. (4.15)

4-104. To pay off $50,000,000 worth of new construction bonds when they come due in 20 years, a water municipality must deposit money into a sinking fund. Payments to the fund will be made quarterly, starting three months from now. If the interest rate for the sinking fund is 8% compounded quarterly, how much will each deposit be? (4.15)

4-105. A health club offers you a special low membership rate of $29 per month for a "guaranteed no price increase" period of 100 months. The manager of the club tells you proudly that "this $29 a month for a lifetime membership is less expensive than a major medical treatment for heart disease costing $4,000 one hundred months from now." If your personal interest rate is 9% (APR) compounded monthly, is the manager correct in his statement? (4.15)

4-106. A light-duty pickup truck has a manufacturer's suggested retail price (MSRP) of $14,000 on its window. After haggling with the salesperson for several days, the prospective buyer is offered the following deal: "You pay a $1,238 down payment now and $249 each month thereafter for 39 months and the truck will be yours." The APR at this dealership is 2.4% compounded monthly. How good a deal is this relative to the MSRP? (4.15)

4-107. You borrow $10,000 from a bank for three years at an annual interest rate, or annual percentage rate (APR), of 12%. Monthly payments will be made until all the principal and interest have been repaid. (4.14, 4.15)

a. What is your monthly payment?

b. If you must pay two points up front, meaning that you get only $9,800 from the bank, what is your true APR on the loan?

4-108. A young couple has decided to set aside some money to fund their 5-year-old daughter's college education. They want to deposit the sum in an account that earns 8% compounded continuously. Beginning when she turns 6 and ending when she turns 21, how much should they deposit on each birthday if she is to get $28,000 on each birthday from her 21st to her 24th? (4.33)

4-109. The longer a loan schedule lasts, the more interest you will pay. To illustrate, a $20,000-car-loan at 9% APR (compounded monthly) for three years (36 monthly payments) will incur total interest of $2,896. But a $20,000 car loan at 9% APR over six years (72 payments) will have a total interest cost of $5,920. (4.15)

a. Verify that the difference in total interest of $3,024 is correct.

b. Why would you be willing to pay this extra interest?

4-110. On January 1, 2005, a person's savings account was worth $200,000. Every month thereafter, this person makes a cash contribution of $676 to the account. If the fund is expected to be worth $400,000 on January 1, 2010, what annual rate of interest is being earned on this fund? (4.15)

4-111. Suppose you owe $1,100 on your credit card. The annual percentage rate (APR) is 18%, compounded monthly. The credit card company says your minimum monthly payment is $19.80. (4.15)

a. If you make only this minimum payment, how long will it take for you to repay the $1,100 balance (assuming no more charges are made)?

b. If you make the minimum payment plus $10 extra each month (for a total of $29.80), how long will it take to repay the $1,100 balance?

c. Compare the total interest paid in Part (a) with the total interest paid in Part (b).

4-112. A student borrowed $6,000, which he will repay in 30 equal monthly instalments. After his 25th payment he wants to pay off the reminder of the loan in a single payment at 15% interest compounded monthly. What is the amount of the payment? **(4.6)**

4-113. A nominal interest rate of 11.333%, continuously compounded, yields an effective annual interest rate of how much? **(4.16)**

4-114. A bank offers a nominal interest rate (APR) of 6%, continuously compounded. What is the effective interest rate? **(4.16)**

4-115. If a nominal interest rate of 8% is compounded continuously, determine the unknown quantity in each of the following situations: **(4.16)**

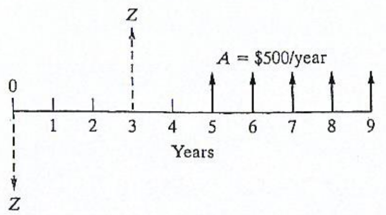
a. What uniform EOY amount for 10 years is equivalent to $8,000 at EOY 10?

b. What is the present equivalent value of $1,000 per year for 12 years?

c. What is the future equivalent at the end of the sixth year of $243 payments made every six months during the six years? The first payment occurs six months from the present and the last occurs at the end of the sixth year.

d. Find the equivalent lump-sum amount at EOY nine when $P_0 = \$1,000$.

4-116. Find the value of the unknown quantity Z in the following diagram, such that the equivalent cash outflow equals the equivalent cash inflows when $\underline{r}$ = 20% compounded continuously: **(4.16)**



4-117. Juan deposits $5,000 into a savings account that pays 7.2% per year, continuously compounded. What is the effective annual interest rate? Determine the value of his account at the end of two years. **(4.16)**

4-118. A man deposited $10,000 in a savings account when his son was born. The nominal interest rate was 8% per year, compounded continuously. On the son's 18th birthday, the accumulated sum is withdrawn from the account. How much will this accumulated amount be? **(4.16)**

4-119. A person needs $18,000 immediately as a down payment on a new home. Suppose that she can borrow this money from her company credit union. She will be required to repay the loan in equal payments *made every six months* over the next 12 years. The annual interest rate being charged is 10% compounded continuously. What is the amount of each payment? **(4.16)**

4-120. What is the present worth of a series of equal year-end payments of $1,500 each for 10 years if the interest rate is 10.8% compounded continuously? **(4.33)**

4-121. Jim is a farsighted 20-year-old who wants to have $5 million saved by the time he's 30. If he can earn 5% on his funds, how much he should invest today to reach his goal? **(4.4)**

4-122. Many persons prepare for retirement by making monthly contributions to a savings program. Suppose that $2,000 is set aside each year and invested in a savings account that pays 10% interest per year, compounded continuously. **(4.16)**

a. Determine the accumulated savings in this account at the end of 30 years.

b. In Part (a), suppose that an annuity will be withdrawn from savings that have been accumulated at the EOY 30. The annuity will extend from the EOY 31 to the EOY 40. What is the value of this annuity if the interest rate and compounding frequency in Part (a) do not change?

4-123. Indicate whether each of the following statements is true (T) or false (F). **(all sections)**

a. T F Interest is money paid for the use of equity capital.

b. T F $(A/F, i\%, N) = (A/P, i\%, N) + i$.

c. T F Simple interest ignores the time value of money principle.

d. T F Cash-flow diagrams are analogous to free-body diagrams for mechanics problems.

e. T F $1,791 10 years from now is equivalent to $900 now if the interest rate equals 8% per year.

f. T F It is always true that $i > r$ when $M \geq 2$.

g. T F Suppose that a lump sum of $1,000 is invested at $\underline{r}$ = 10% for eight years. The future equivalent is greater for daily compounding than it is for continuous compounding.

h. T F For a fixed amount, F dollars, that is received at EOY N, the "A equivalent" increases as the interest rate increases.

i. T F For a specified value of F at EOY N, P at time zero will be larger for $\underline{r}$ = 10% per year than it will be for r = 10% per year, compounded monthly.

4-124. Mark each statement true (T) or false (F), and fill in the blanks in Part (e). **(all sections)**

a. T F The nominal interest rate will always be less than the effective interest rate when r = 10% and $M = \infty$.

b. T F A certain loan involves monthly repayments of $185 over a 24-month period. If $\underline{r}$ = 10% per year, more than half of the principal is still owed on this loan after the 10th monthly payment is made.

c. T F $1,791 in 10 years is equivalent to $900 now if the nominal interest rate is 8% compounded semiannually.

d. T F The $(P/A, i\%, N)$ factor equals $N \cdot (P/F, i\%, 1)$.

e. Fill in the missing interest factor:

i. $(P/A, i\%, N)(\text{———}) = (F/A, i\%, N)$.

ii. $(A/G, i\%, N)(P/A, i\%, N) = (\text{———})$.

4-125. A mutual fund investment is expected to earn 11% per year for the next 25 years. If inflation will average 3% per year during this 25-year period of time, what is the compounded value (in today's dollars) of this savings vehicle when $10,000 is invested now? **(4.6)**

4-126. Javier just bought a condominium in College-town, USA. His $100,000 mortgage is 6% compounded monthly, and Javier will make monthly payments on his loan for 30 years. In addition, property taxes and title insurance amount to $400 per month. **(4.15)**

a. What is the total mortgage-related amount of Javier's monthly condo payment?

b. Develop an estimate of Javier's total monthly expenses (maintenance, utilities, and so on) for his condominium.

c. If Javier qualifies for a 15-year mortgage having an APR of 5.8% compounded monthly, what will his monthly mortgage payment be (there will be 180 payments)?

4-127. Analyze the truth of this statement, assuming you are 20 years old: "For every five years that you wait to start accumulating money for your retirement, it takes twice as much savings per year to catch up." Base your analysis on the target of having $1,000,000 when you retire at age 60. Be sure to state your assumptions. **(4.7)**

4-128. A mortgage company advertises that their 6% APR is an effective annual rate of 6.58% with monthly payments and compounding. Let's assume this is made possible by paying points on a mortgage (one point is 1% of the loan amount). How many points are being paid up front on a $100,000 mortgage loan over 15 years to arrive at an effective interest rate of 6.58%? **(4.15)**

Spreadsheet Exercises

4-129. Create a spreadsheet that duplicates Table 4-1. Make it flexible enough that you can investigate the impact of different interest rates and principal loan amounts without changing the structure of the spreadsheet. **(4.7)**

4-130. A $15,000 investment is to be made with anticipated annual returns as shown in the spreadsheet in Figure P4-130. If the investor's time value of money is 10% per year, what should be entered in cells B11, B12, and B13 to obtain present, annual, and future equivalent values for the investment? **(4.10)**

	A	B
1	EOY	Cash flow
2	0	−$15,000
3	1	$2,000
4	2	$2,500
5	3	$3,000
6	4	$3,500
7	5	$4,000
8	6	$4,000
9	7	$4,000
10	8	$4,000
11	P	
12	A	
13	F	

Figure P4-130 Spreadsheet for Problem 4-130

4-131. Refer to Example 4-23. Suppose the cash-flow sequence continues for 10 years (instead of four). Determine the new values of P, A, and F. **(4.12)**

4-132. Higher interest rates won't cost you as much to drive a car as do higher gasoline prices. This is because automobile loans have payment schedules that are only modestly impacted by Federal Reserve Board interest-rate increases. Create a spreadsheet to compare the difference in monthly payments for a $25,000 loan having 60 monthly payments for a select number of interest rates. Use 3% as the base APR and go as high as an APR of 12%. *Challenge:* Make your spreadsheet flexible enough to be able to look at the impact of the different interest rates for different loan amounts and different repayment periods. **(4.15)**

Case Study Exercises

4-133. After Enrico's car is paid off, he plans to continue setting aside the amount of his car payment to accumulate funds for the car's replacement. If he invests this amount at a rate of 3% compounded monthly, how much will he have saved by the end of the initial 10-year period? **(4.17)**

4-134. Enrico has planned to have $40,000 at the end of 10 years to place a down payment on a condo. Property taxes and insurance can be as much as 30% of the monthly principal and interest payment (i.e., for a principal and interest payment of $1,000, taxes and insurance would be an additional $300). What is the maximum purchase price he can afford if he'd like to keep his housing costs at $950 per month? **(4.17)**

4-135. If Enrico is more daring with his retirement investment savings and feels he can average 10% per year, how much will he have accumulated for retirement at the end of the 10-year period? **(4.17)**

FE Practice Problems

4-136. If you borrow $5,000 at 8% simple interest per year for ten years, how much will you have to repay at the end of ten years? **(4.2)**

(a) $3,000 (b) $4,511 (c) $6,000
(d) $8,000 (e) $9,000

4-137. When you were born, your grandfather established a trust fund for you in the Cayman Islands. The account has been earning interest at the rate of 10% per year. If this account will be worth $100,000 on your 25th birthday, how much did your grandfather deposit on the day you were born? **(4.6)**

(a) $4,000 (b) $9,230 (c) $10,000
(d) $10,150 (e) $10,740

4-138. Every year you deposit $2,500 into an account that earns 5% interest per year. What will be the balance of your account immediately after the 20th deposit? **(4.7)**

(a) $44,793 (b) $60,000 (c) $77,385
(d) $81,136 (e) $82,665

4-139. Your monthly mortgage payment (principal plus interest) is $1,200. If you have a 15-year loan with a fixed interest rate of 2% per month, how much did you borrow from the bank to purchase your house? Select the closest answer. **(4.7)**

(a) $154,00 (b) $180,00 (c) $250,00
(d) $300,00 (e) $540,00

4-140. Consider the following sequence of year-end cash flows:

EOY	1	2	3	4	5
Cash Flow	$8,000	$15,000	$22,000	$29,000	$36,000

What is the uniform annual equivalent if the interest rate is 12% per year? **(4.11)**

(a) $20,422 (b) $17,511 (c) $23,204
(d) $22,000 (e) $12,422

4-141. A cash flow at time zero (now) of $9,982 is equivalent to another cash flow that is an EOY annuity of $2,500 over five years. Each of these two cash-flow series is equivalent to a third series, which is a uniform gradient series. What is the value of G for this third series over the same five-year time interval? **(4.11)**

(a) $994 (b) $1,150 (c) $1,250
(d) $1,354 (e) Not enough information given

4-142. Bill Mitselfik borrowed $10,000 to be repaid in quarterly installments over the next five years. The interest rate he is being charged is 12% per year compounded quarterly. What is his quarterly payment? **(4.15)**

(a) $400 (b) $550 (c) $650
(d) $800

4-143. Sixty monthly deposits are made into an account paying 6% nominal interest compounded monthly. If the objective of these deposits is to accumulate $100,000 by the end of the fifth year, what is the amount of each deposit? **(4.15)**

(a) $1,930 (b) $1,478 (c) $1,667
(d) $1,430 (e) $1,695

4-144. What is the principal remaining after 20 monthly payments have been made on a $20,000 five-year loan? The annual interest rate is 12% nominal compounded monthly. **(4.15)**

(a) $10,224 (b) $13,333 (c) $14,579
(d) $16,073 (e) $17,094

4-145. If you borrow $5,000 to buy a car at 6% compounded monthly, to be repaid over the next five years, what is your monthly payment? **(4.15)**

(a) $81.7 (b) $96.5 (c) $112
(d) $89 (e) $104

4-146. The effective annual interest rate is given to be 19.2%. What is the nominal interest rate per year (r) if continuous compounding is being used? Choose the closest answer below. **(4.16)**

(a) 19.83% (b) 18.55% (c) 17.56%
(d) 16.90%

4-147. A bank advertises mortgages at 12% compounded continuously. What is the effective annual interest? **(4.16)**

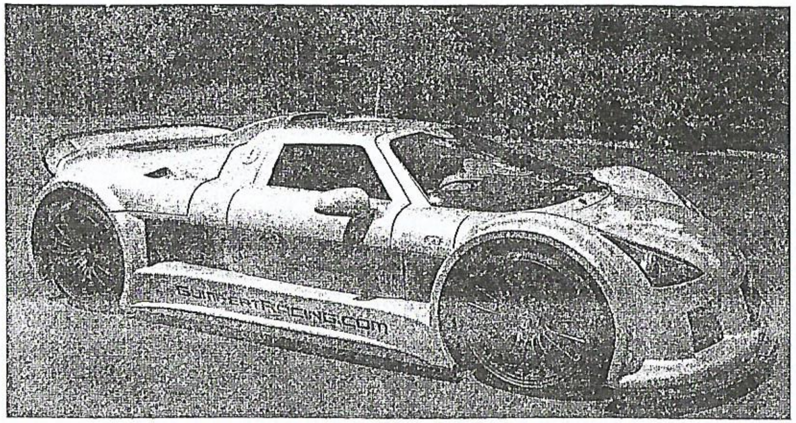
(a) 12.36% (b) 12.55% (c) 12.75%
(d) 12.68% (e) 12.00%

4-148. If you invest $7,000 at 12% compounded continuously, how much would it be worth in three years? **(4.16)**

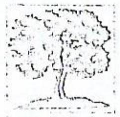
(a) $9,449 (b) $4,883 (c) $10,033
(d) $9,834 (e) $2,520

CHAPTER 5

Evaluating a Single Project



The objective of Chapter 5 is to discuss and critique contemporary methods for determining project profitability.



Carbon Fibers versus Metal

There is a rule-of-thumb that "if the weight of an automobile can be reduced by 10%, then 6% of the annual cost of gasoline can be saved." Light weight and high strength carbon fibers costing about \$15–\$20 per pound are currently being considered to replace the metal in automobile and aerospace applications. (The objective of recent research is to reduce this cost to \$5 per pound.) Engineers believe they can economically reduce the weight of an automobile by substituting carbon fibers for metal to save 20% to 30% on fuel consumption each year. Other structures such as stronger wind turbines can also be built with light weight carbon fibers. After working through this chapter, you will be able to evaluate the economic trade-off between annual fuel savings and up-front cost of carbon fibers and to determine whether it is a smart trade-off.

The final test of any system is, does it pay?

—Frederick W. Taylor (1912)

5.1 Introduction

All engineering economy studies of capital projects should consider the return that a given project will or should produce. A basic question this book addresses is whether a proposed capital investment and its associated expenditures can be recovered by revenue (or savings) over time *in addition to* a return on the capital that is sufficiently attractive in view of the risks involved and the potential alternative uses. The interest and money–time relationships discussed in Chapter 4 emerge as essential ingredients in answering this question, and they are applied to many different types of problems in this chapter.

Because patterns of capital investment, revenue (or savings) cash flows, and expense cash flows can be quite different in various projects, there is no single method for performing engineering economic analyses that is ideal for all cases. Consequently, several methods are commonly used. A project focus will be taken as we introduce ways of gauging profitability.

In this chapter, we concentrate on the correct use of five methods for evaluating the economic profitability of a single proposed problem solution (i.e., *alternative*).* Later, in Chapter 6, multiple alternatives are evaluated. The five methods described in Chapter 5 are Present Worth (PW), Future Worth (FW), Annual Worth (AW), Internal Rate of Return (IRR), and External Rate of Return (ERR). The first three methods convert cash flows resulting from a proposed problem solution into their equivalent worth at some point (or points) in time by using an interest rate known as the *Minimum Attractive Rate of Return (MARR)*. The concept of a MARR, as well as the determination of its value, is discussed in the next section. The IRR and ERR methods compute annual rates of profit, or returns, resulting from an investment and are then compared to the MARR.

The payback period is also discussed briefly in this chapter. The payback period is a measure of the *speed* with which an investment is recovered by the cash inflows it produces. This measure, in its most common form, ignores time value of money principles. For this reason, the payback method is often used to supplement information produced by the five primary methods featured in this chapter.

Unless otherwise specified, the end-of-period cash-flow convention and discrete compounding of interest are used throughout this and subsequent chapters. A planning horizon, or study (analysis) period, of N compounding periods (usually years) is used to evaluate prospective investments throughout the remainder of the book.

* The analysis of engineering projects using the benefit–cost ratio method is discussed in Chapter 10.

5.2 Determining the Minimum Attractive Rate of Return (MARR)

The Minimum Attractive Rate of Return (MARR) is usually a policy issue resolved by the top management of an organization in view of numerous considerations. Among these considerations are the following:

1. The amount of money available for investment, and the source and cost of these funds (i.e., equity funds or borrowed funds)
2. The number of good projects available for investment and their purpose (i.e., whether they sustain present operations and are *essential*, or whether they expand on present operations and are *elective*)
3. The amount of perceived risk associated with investment opportunities available to the firm and the estimated cost of administering projects over short planning horizons versus long planning horizons
4. The type of organization involved (i.e., government, public utility, or private industry)

In theory, the MARR, which is sometimes called the *hurdle rate*, should be chosen to maximize the economic well-being of an organization, subject to the types of considerations just listed. How an individual firm accomplishes this in practice is far from clear-cut and is frequently the subject of discussion. One popular approach to establishing a MARR involves the *opportunity cost* viewpoint described in Chapter 2, and it results from the phenomenon of *capital rationing*. This situation may arise when the amount of available capital is insufficient to sponsor all worthy investment opportunities. The subject of capital rationing is covered in Chapter 13.

A simple example of capital rationing is given in Figure 5-1, where the cumulative investment requirements of seven acceptable projects are plotted against the prospective annual rate of profit of each. Figure 5-1 shows a limit of $600 million on available capital. In view of this limitation, the last funded project would be *E*, with a prospective rate of profit of 19% per year, and the best *rejected* project is *F*. In this case, the MARR by the opportunity cost principle would be 16% per year. By *not* being able to invest in project *F*, the firm would presumably be forfeiting the chance to realize a 16% annual return. As the amount of investment capital and opportunities available change over time, the firm's MARR will also change.*

Superimposed on Figure 5-1 is the approximate cost of obtaining the $600 million, illustrating that project *E* is acceptable only as long as its annual rate of profit exceeds the cost of raising the last $100 million. As shown in Figure 5-1, the cost of capital will tend to increase gradually as larger sums of money are acquired through increased borrowing (debt) or new issuances of common stock (equity). Determining the MARR is discussed further in Chapter 13.

* As we shall see in Chapter 11, the selection of a project may be relatively insensitive to the choice of a value for the MARR. Revenue estimates, for example, are much more important to the selection of the most profitable investment.

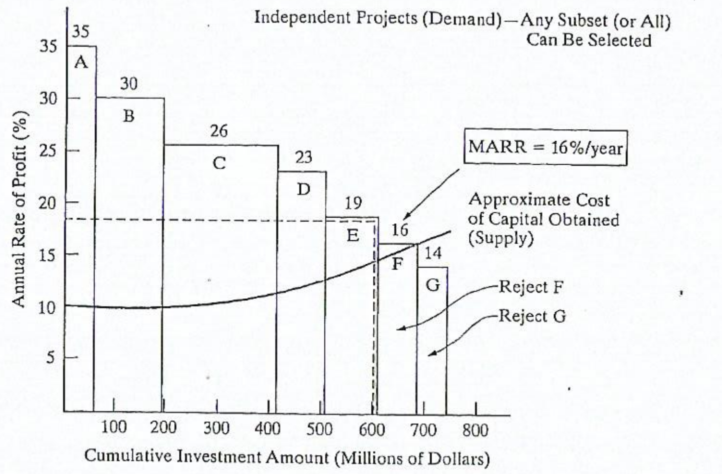


Figure 5-1 Determination of the MARR Based on the Opportunity Cost Viewpoint (A popular measure of annual rate of profit is "Internal Rate of Return," discussed later in this chapter.)

5.3 The Present Worth Method

The PW method is based on the concept of equivalent worth of all cash flows relative to some base or beginning point in time called the present. That is, all cash inflows and outflows are discounted to the present point in time at an interest rate that is generally the MARR. A positive PW for an investment project is a dollar amount of profit over the minimum amount required by investors. It is assumed that cash generated by the alternative is available for other uses that earn interest at a rate equal to the MARR.

To find the PW as a function of $i\%$ (per interest period) of a series of cash inflows and outflows, it is necessary to discount future amounts to the present by using the interest rate over the appropriate study period (years, for example) in the following manner:

$$\begin{aligned} \text{PW}(i\%) &= F_0(1+i)^0 + F_1(1+i)^{-1} + F_2(1+i)^{-2} \\ &\quad + \cdots + F_k(1+i)^{-k} + \cdots + F_N(1+i)^{-N} \\ &= \sum_{k=0}^{N} F_k(1+i)^{-k}. \end{aligned} \quad (5\text{-}1)$$

Here, i = effective interest rate, or MARR, per compounding period;
k = index for each compounding period ($0 \le k \le N$);
F_k = future cash flow at the end of period k;
N = number of compounding periods in the planning horizon (i.e., study period).

The relationship given in Equation (5-1) is based on the assumption of a *constant interest rate* throughout the life of a particular project. If the interest rate is assumed to change, the PW must be computed in two or more steps, as was illustrated in Chapter 4.

To apply the PW method of determining a project's economic worthiness, we simply compute the present equivalent of all cash flows using the MARR as the interest rate. If the present worth is greater than or equal to zero, the project is acceptable.

PW Decision Rule: If PW (i = MARR) ≥ 0, the project is economically justified.

It is important to observe that the higher the interest rate and the farther into the future a cash flow occurs, the lower its PW is. This is shown graphically in Figure 5-2. The PW of $1,000 10 years from now is $613.90 when $i = 5\%$ per year. However, if $i = 10\%$, that same $1,000 is only worth $385.50 now.

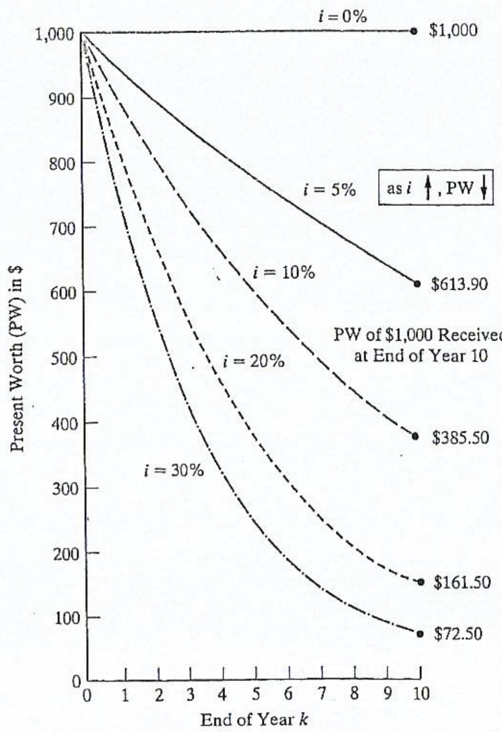
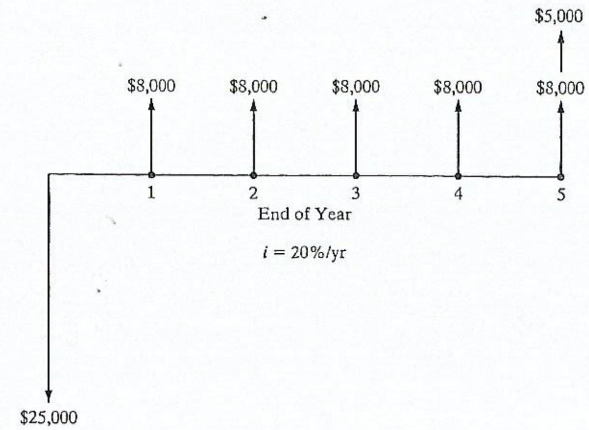


Figure 5-2 PW of $1,000 Received at the End of Year k at an Interest Rate of i% per Year

EXAMPLE 5-1

Evaluation of New Equipment Purchase Using PW

A piece of new equipment has been proposed by engineers to increase the productivity of a certain manual welding operation. The investment cost is $25,000, and the equipment will have a market value of $5,000 at the end of a study period of five years. Increased productivity attributable to the equipment will amount to $8,000 per year after extra operating costs have been subtracted from the revenue generated by the additional production. A cash-flow diagram for this investment opportunity is given below. If the firm's MARR is 20% per year, is this proposal a sound one? Use the PW method.



Solution

$$\text{PW} = \text{PW of cash inflows} - \text{PW of cash outflows},$$

or

$$\text{PW}(20\%) = \$8{,}000(P/A, 20\%, 5) + \$5{,}000(P/F, 20\%, 5) - \$25{,}000$$
$$= \$934.29.$$

Because PW(20%) ≥ 0, this equipment is economically justified.

The MARR in Example 5-1 (and in other examples throughout this chapter) is to be interpreted as an effective interest rate (i). Here, $i = 20\%$ per year. Cash flows are discrete, end-of-year (EOY) amounts. If *continuous compounding* had been specified for a nominal interest rate (r) of 20% per year, the PW would have been calculated by using the interest factors presented in Appendix D:

$$\begin{aligned}PW(\underline{r} = 20\%) &= -\$25{,}000 + \$8{,}000(P/A, \underline{r} = 20\%, 5)\\ &\quad + \$5{,}000(P/F, \underline{r} = 20\%, 5)\\ &= -\$25{,}000 + \$8{,}000(2.8551) + \$5{,}000(0.3679)\\ &= -\$319.60.\end{aligned}$$

Consequently, with continuous compounding, the equipment would not be economically justifiable. The reason is that the higher effective annual interest rate ($e^{0.20} - 1 = 0.2214$) reduces the PW of future positive cash flows but does not affect the PW of the capital invested at the beginning of year one.

EXAMPLE 5-2

Present Worth of a Space-Heating System

A retrofitted space-heating system is being considered for a small office building. The system can be purchased and installed for \$110,000, and it will save an estimated 300,000 kilowatt-hours (kWh) of electric power each year over a six-year period. A kilowatt-hour of electricity costs \$0.10, and the company uses a MARR of 15% per year in its economic evaluations of refurbished systems. The market value of the system will be \$8,000 at the end of six years, and additional annual operating and maintenance expenses are negligible. Use the PW method to determine whether this system should be installed.

Solution

To find the PW of the proposed heating system, we need to find the present equivalent of all associated cash flows. The estimated annual savings in electrical power is worth 300,000 kWh × \$0.10/kWh = \$30,000 per year. At a MARR of 15%, we get

$$\begin{aligned}PW(15\%) &= -\$110{,}000 + \$30{,}000\ (P/A, 15\%, 6) + \$8{,}000\ (P/F, 15\%, 6)\\ &= -\$110{,}000 + \$30{,}000(3.7845) + \$8{,}000(0.4323)\\ &= \$6{,}993.40.\end{aligned}$$

Since $PW(15\%) \geq 0$, we conclude that the retrofitted space-heating system should be installed.

Now that we know how to apply the PW method, we can use it to evaluate the economic advisability of a solar-powered cooling and heating system. Should a homeowner's incremental investment of \$10,000 be traded off for energy savings of \$130 per month? Let's assume a MARR of 0.5% per month and a 20-year useful life of the solar-powered system. In this case, the PW of the system is

$$\begin{aligned}PW &= -\$10{,}000 + \$130(P/A, 0.5\%, \text{ per month, } 240 \text{ months})\\ &= -\$10{,}000 + \$130(139.5808)\\ &= \$8{,}145.50.\end{aligned}$$

The positive-valued PW signals a favorable investment. Additionally, 13 tons/year × 20 years = 260 tons of carbon dioxide will be avoided. Can you rework this problem when the MARR is 1% per month? Is the system still a judicious choice?

5.3.1 Assumptions of the PW Method

There are several noteworthy assumptions that we make when using PW to model the wealth-creating promise of a capital investment opportunity. First, it is assumed that we know the future with certainty (we don't live in a certain world!). For example, we presume to know with certainty future interest rates and other factors. Second, it is assumed we can borrow and lend money at the same interest rate (i.e., capital markets are perfect). Regrettably, the real world has neither certainty nor perfect (*frictionless*, e.g., no taxes and/or commissions) capital markets.

The PW (and FW and AW, to follow) model is built on these seemingly restrictive assumptions, but it is cost-beneficial in the sense that the cost of using the PW model is less than the benefits of improved decisions resulting from PW analysis. More sophisticated models exist, but they usually do not reverse decisions made with the PW model. Therefore, our goal is to cost-beneficially recommend capital investments that maximize the wealth of a firm to its owners (i.e., stockholders). A positive-valued PW (and FW and AW) means that accepting a project will increase the worth, or value, of the firm.

5.3.2 Bond Value

A bond is an IOU where you agree to lend the bond issuer money for a specified length of time (say, 10 years). In return, you receive periodic interest payments (e.g., quarterly) from the issuer plus a promise to return the face value of the bond when it matures. A bond provides an excellent example of commercial value as being the PW of the future net cash flows that are expected to be received through ownership of an interest-bearing certificate. Thus, the value of a bond, at any time, is the PW of future cash receipts. For a bond, let

Z = face, or par, value;
C = redemption or disposal price (usually equal to Z);
r = bond rate (nominal interest) per interest period;
N = number of periods before redemption;
i = bond *yield* rate per period;
V_N = value (price) of the bond N interest periods prior to redemption—this is a PW measure of merit.

The owner of a bond is paid two types of payments by the borrower. The first consists of the series of periodic interest payments he or she will receive until the bond is retired. There will be N such payments, each amounting to rZ. These constitute an annuity of N payments. In addition, when the bond is retired or sold, the bondholder will receive a single payment equal in amount to C. The PW of the bond is the sum of PWs of these two types of payments at the bond's yield rate ($i\%$):

$$V_N = C(P/F, i\%, N) + rZ(P/A, i\%, N). \quad (5\text{-}2)$$

The most common situations faced by you as a potential investor in bonds are (1) for a desired yield rate, how much should you be willing to pay for the bond and (2) for a stated purchase price, what will your yield be? Examples 5-3 and 5-4 demonstrate how to solve these types of problems.

EXAMPLE 5-3

Stan Moneymaker Wants to Buy a Bond

Stan Moneymaker has the opportunity to purchase a certain U.S. Treasury bond that matures in eight years and has a face value of $10,000. This means that Stan will receive $10,000 cash when the bond's maturity date is reached. The bond stipulates a fixed nominal interest rate of 8% per year, but interest payments are made to the bondholder every three months; therefore, each payment amounts to 2% of the face value.

Stan would like to earn 10% nominal interest (compounded quarterly) per year on his investment, because interest rates in the economy have risen since the bond was issued. How much should Stan be willing to pay for the bond?

Solution

To establish the value of this bond, in view of the stated conditions, the PW of future cash flows during the next eight years (the study period) must be evaluated. Interest payments are quarterly. Because Stan Moneymaker desires to obtain 10% *nominal interest per year* on the investment, the PW is computed at $i = 10\%/4 = 2.5\%$ per quarter for the remaining $8(4) = 32$ quarters of the bond's life:

$$V_N = \$10{,}000(P/F, 2.5\%, 32) + \$10{,}000(0.02)(P/A, 2.5\%, 32)$$
$$= \$4{,}537.71 + \$4{,}369.84 = \$8{,}907.55.$$

Thus, Stan should pay no more than $8,907.55 when 10% nominal interest per year is desired.

EXAMPLE 5-4

Current Price and Annual Yield of Bond Calculations

A bond with a face value of $5,000 pays interest of 8% per year. This bond will be redeemed at par value at the end of its 20-year life, and the first interest payment is due one year from now.

(a) How much should be paid now for this bond in order to receive a yield of 10% per year on the investment?

(b) If this bond is purchased now for $4,600, what annual yield would the buyer receive?

Solution

(a) By using Equation (5-2), the value of V_N can be determined:

$$V_N = \$5{,}000(P/F, 10\%, 20) + \$5{,}000(0.08)(P/A, 10\%, 20)$$
$$= \$743.00 + \$3{,}405.44 = \$4{,}148.44.$$

(b) Here, we are given $V_N = \$4{,}600$, and we must find the value of $i\%$ in Equation (5-2):

$$\$4{,}600 = \$5{,}000(P/F, i'\%, 20) + \$5{,}000(0.08)(P/A, i'\%, 20).$$

To solve for $i'\%$, we can resort to an iterative trial-and-error procedure (e.g., try 8.5%, 9.0%), to determine that $i'\% = 8.9\%$ per year.

5.3.3 The Capitalized-Worth Method

One special variation of the PW method discussed in Section 5.3 involves determining the PW of all revenues or expenses over an infinite length of time. This is known as the *Capitalized-Worth* (CW) method. If only expenses are considered, results obtained by this method are sometimes referred to as *capitalized cost*. As will be demonstrated in Chapter 6, the CW method is a convenient basis for comparing mutually exclusive alternatives when the period of needed service is indefinitely long.

The CW of a perpetual series of end-of-period uniform payments A, with interest at $i\%$ per period, is $A(P/A, i\%, \infty)$. From the interest formulas, it can be seen that $(P/A, i\%, N) \to 1/i$ as N becomes very large. Thus, $\text{CW} = A/i$ for such a series, as can also be seen from the relation

$$\text{CW}(i\%) = \text{PW}_{N\to\infty} = A(P/A, i\%, \infty) = A\left[\lim_{N\to\infty} \frac{(1+i)^N - 1}{i(1+i)^N}\right] = A\left(\frac{1}{i}\right).$$

Hence, the CW of a project with interest at $i\%$ per year is the annual equivalent of the project over its useful life divided by i (as a decimal).

The AW of a series of payments of amount $X at the end of each kth period with interest at $i\%$ per period is $\$X(A/F, i\%, k)$. The CW of such a series can thus be calculated as $\$X(A/F, i\%, k)/i$.

EXAMPLE 5-5

Determining the Capitalized Worth of a Bridge

A new bridge across the Cumberland River is being planned near a busy highway intersection in the commercial part of a mid-western town. The construction (first) cost of the bridge is $1,900,000 and annual upkeep is estimated to be $25,000. In addition to annual upkeep, major maintenance work is anticipated every eight years at a cost of \$350,000 per occurrence. The town government's MARR is 8% per year.

(a) For this problem, what analysis period (N) is, practically speaking, defined as forever?

(b) If the bridge has an expected life of 50 years, what is the capitalized worth (CW) of the bridge over a 100-year study period?

Solution

(a) A practical approximation of "forever" (infinity) is dependent on the interest rate. By examining the $(A/P, i\%, N)$ factor as N increases in the Appendix C tables, we observe that this factor approaches a value of i as N becomes large. For $i = 8\%$ (Table C-11), the $(A/P, 8\%, 100)$ factor is 0.08. So $N = 100$ years is, for practical purposes, "forever" in this example.

(b) The CW is determined as follows:

$$\begin{aligned} CW(8\%) &= -\$1{,}900{,}000 - \$1{,}900{,}000\ (P/F, 8\%, 50) \\ &\quad - [\$350{,}000\ (A/F, 8\%, 8)]/0.08 - \$25{,}000/0.08. \end{aligned}$$

The CW turns out to be −\$2,664,220 over a 100-year study period, assuming the bridge is replaced at the end of year 50 for \$1,900,000.

5.4 The Future Worth Method

Because a primary objective of all time value of money methods is to maximize the future wealth of the owners of a firm, the economic information provided by the FW method is very useful in capital investment decision situations. The FW is based on the equivalent worth of all cash inflows and outflows at the end of the planning horizon (study period) at an interest rate that is generally the MARR. Also, the FW of a project is equivalent to its PW; that is, $FW = PW(F/P, i\%, N)$. If $FW \geq 0$ for a project, it would be economically justified.

FW Decision Rule: If FW (i = MARR) ≥ 0, the project is economically justified.

Equation (5-3) summarizes the general calculations necessary to determine a project's FW:

$$\begin{aligned} FW(i\%) &= F_0(1+i)^N + F_1(1+i)^{N-1} + \cdots + F_N(1+i)^0 \\ &= \sum_{k=0}^{N} F_k(1+i)^{N-k}. \end{aligned} \tag{5-3}$$

EXAMPLE 5-6 **The Relationship between FW and PW**

Evaluate the FW of the potential improvement project described in Example 5-1. Show the relationship between FW and PW for this example.

Solution

$$\begin{aligned} FW(20\%) &= -\$25{,}000(F/P, 20\%, 5) \\ &\quad + \$8{,}000(F/A, 20\%, 5) + \$5{,}000 \\ &= \$2{,}324.80. \end{aligned}$$

Again, the project is shown to be a good investment ($FW \geq 0$). The PW is a multiple of the equivalent FW value:

$$PW(20\%) = \$2{,}324.80(P/F, 20\%, 5) = \$934.29.$$

To this point, the PW and FW methods have used a known and constant MARR over the study period. Each method produces a measure of merit expressed in dollars and is equivalent to the other. The difference in economic information provided is relative to the point in time used (i.e., the present for the PW versus the future, or end of the study period, for the FW).

EXAMPLE 5-7 **Sensitivity Analysis Using FW (Example 5-2 Revisited)**

In Example 5-2, the \$110,000 retrofitted space-heating system was projected to save \$30,000 per year in electrical power and be worth \$8,000 at the end of the six-year study period. Use the FW method to determine whether the project is still economically justified if the system has zero market value after six years. The MARR is 15% per year.

Solution

In this example, we need to find the future equivalent of the \$110,000 investment and the \$30,000 annual savings at an interest rate of 15% per year.

$$\begin{aligned} FW(15\%) &= -\$110{,}000\ (F/P, 15\%, 6) + \$30{,}000\ (F/A, 15\%, 6) \\ &= -\$110{,}000\ (2.3131) + \$30{,}000\ (8.7537) \\ &= \$8{,}170. \end{aligned}$$

The heating system is still a profitable project ($FW \geq 0$) even if it has no market value at the end of the study period.

5.5 The Annual Worth Method

The AW of a project is an equal annual series of dollar amounts, for a stated study period, that is *equivalent* to the cash inflows and outflows at an interest rate that is generally the MARR. Hence, the AW of a project is annual equivalent revenues or savings ($\underline{R}$) minus annual equivalent expenses ($\underline{E}$), less its annual equivalent capital recovery (CR) amount, which is defined in Equation (5-5). An annual equivalent value of $\underline{R}$, $\underline{E}$, and CR is computed for the study period, N, which is usually in years. In equation form, the AW, which is a function of $i\%$, is

$$\text{AW}(i\%) = \underline{R} - \underline{E} - \text{CR}(i\%). \qquad (5\text{-}4)$$

Also, we need to notice that the AW of a project is equivalent to its PW and FW. That is, AW = PW(A/P, $i\%$, N), and AW = FW(A/F, $i\%$, N). Hence, it can be easily computed for a project from these other equivalent values.

As long as the AW evaluated at the MARR is greater than or equal to zero, the project is economically attractive; otherwise, it is not. An AW of zero means that an annual return exactly equal to the MARR has been earned. Many decision makers prefer the AW method because it is relatively easy to interpret when they are accustomed to working with annual income statements and cash-flow summaries.

AW Decision Rule: If AW (i = MARR) ≥ 0, the project is economically justified.

The CR amount for a project is the equivalent uniform annual *cost* of the capital invested. It is an annual amount that covers the following two items:

1. Loss in value of the asset
2. Interest on invested capital (i.e., at the MARR)

As an example, consider a device that will cost \$10,000, last five years, and have a salvage (market) value of \$2,000. Thus, the loss in value of this asset over five years is \$8,000. Additionally, the MARR is 10% per year.

It can be shown that, no matter which method of calculating an asset's loss in value over time is used, the equivalent annual CR amount is the same. For example, if a uniform loss in value is assumed (\$8,000/5 years = \$1,600 per year), the equivalent annual CR amount is calculated to be \$2,310, as shown in Table 5-1.

There are several convenient formulas by which the CR amount (cost) may be calculated to obtain the result in Table 5-1. Probably the easiest formula to understand involves finding the annual equivalent of the initial capital investment and then subtracting the annual equivalent of the salvage value.* Thus,

$$\text{CR}(i\%) = I(A/P, i\%, N) - S(A/F, i\%, N), \qquad (5\text{-}5)$$

* The following two equations are alternative ways of calculating the CR amount:

$$\text{CR}(i\%) = (I - S)(A/F, i\%, N) + I(i\%);$$

$$\text{CR}(i\%) = (I - S)(A/P, i\%, N) + S(i\%).$$

It is left as a student exercise to show that the above equations are equivalent to Equation (5-5).

Calculation of Equivalent Annual CR Amount

Year	Value of Investment at Beginning of Year[a]	Uniform Loss in Value	Interest on Beginning-of-Year Investment at $i = 10\%$	CR Amount for Year	PW of CR Amount at $i = 10\%$
1	\$10,000	\$1,600	\$1,000	\$2,600	\$2,600(P/F, 10%, 1) = \$2,364
2	8,400	1,600	840	2,440	\$2,440(P/F, 10%, 2) = \$2,016
3	6,800	1,600	680	2,280	\$2,280(P/F, 10%, 3) = \$1,713
4	5,200	1,600	520	2,120	\$2,120(P/F, 10%, 4) = \$1,448
5	3,600	1,600	360	1,960	\$1,960(P/F, 10%, 5) = \$1,217
					\$8,758

CR(10%) = \$8,758(A/P, 10%, 5) = \$2,310.

[a] This is also referred to later as the *beginning-of-year unrecovered investment.*

where I = initial investment for the project;*
S = salvage (market) value at the end of the study period;
N = project study period.

Thus, by substituting the CR($i\%$) expression of Equation (5-5) into the AW expression, Equation (5-4) becomes

$$\text{AW}(i\%) = \underline{R} - \underline{E} - I(A/P, i\%, N) + S(A/F, i\%, N).$$

When Equation (5-5) is applied to the example in Table 5-1, the CR cost is

$$\text{CR}(10\%) = \$10{,}000(A/P, 10\%, 5) - \$2{,}000(A/F, 10\%, 5)$$
$$= \$10{,}000(0.2638) - \$2{,}000(0.1638) = \$2{,}310.$$

EXAMPLE 5-8

Using AW to Evaluate the Purchase of New Equipment (Example 5-1 Revisited)

By using the AW method and Equation (5-4), determine whether the equipment described in Example 5-1 should be recommended.

Solution

The AW Method applied to Example 5-1 yields the following:

$$\text{AW}(20\%) = \overbrace{\$8{,}000}^{\underline{R}-\underline{E}} - \overbrace{[\$25{,}000(A/P, 20\%, 5) - \$5{,}000(A/F, 20\%, 5)]}^{\text{CR amount [Equation (5-5)]}}$$
$$= \$8{,}000 - \$8{,}359.50 + \$671.90$$
$$= \$312.40.$$

* In some cases, the investment will be spread over several periods. In such situations, I is the PW of all investment amounts.

Because its AW(20%) is positive, the equipment more than pays for itself over a period of five years, while earning a 20% return per year on the unrecovered investment. In fact, the annual equivalent "surplus" is \$312.40, which means that the equipment provided more than a 20% return on beginning-of-year unrecovered investment. This piece of equipment should be recommended as an attractive investment opportunity. Also, we can confirm that the AW(20%) is equivalent to PW(20%) = \$934.29 in Example 5-1 and FW(20%) = \$2,324.80 in Example 5-6. That is,

$$\text{AW}(20\%) = \$934.29(A/P, 20\%, 5) = \$312.40, \text{ and also}$$

$$\text{AW}(20\%) = \$2{,}324.80(A/F, 20\%, 5) = \$312.40.$$

When revenues are *absent* in Equation (5-4), we designate this metric as EUAC(i%) and call it "equivalent uniform annual cost." A low-valued EUAC(i%) is preferred to a high-valued EUAC(i%).

EXAMPLE 5-9

Equivalent Uniform Annual Cost of a Corporate Jet

A corporate jet costs \$1,350,000 and will incur \$200,000 per year in fixed costs (maintenance, licenses, insurance, and hangar rental) and \$277 per hour in variable costs (fuel, pilot expense, etc.). The jet will be operated for 1,200 hours per year for five years and then sold for \$650,000. The MARR is 15% per year.

(a) Determine the capital recovery cost of the jet.
(b) What is the EUAC of the jet?

Solution

(a) $\text{CR} = \$1{,}350{,}000\,(A/P,\ 15\%,\ 5) - \$650{,}000\,(A/F,\ 15\%,\ 5) = \$306{,}310.$
(b) The total annual expense for the jet is the sum of the fixed costs and the variable costs.

$$\underline{E} = \$200{,}000 + (1{,}200 \text{ hours})(\$277/\text{hour}) = \$532{,}400$$

$$\text{EUAC}(15\%) = \$532{,}400 + \$306{,}310 = \$838{,}710$$

EXAMPLE 5-10

Determination of Annual Savings by Using the AW Method (Example 5-2 Revisited)

Consider the retrofitted space-heating system described in Example 5-2. Given the investment of \$110,000 and market value of \$8,000 at the end of the six-year study period, what is the minimum annual electrical power savings (in kWh) required to make this project economically acceptable? The MARR = 15% per year and electricity costs \$0.10 per kWh.

Solution

To make this project acceptable, the annual power savings must be at least as great as the annual CR amount. Using Equation (5-5),

$$\text{CR} = \$110{,}000(A/P,\ 15\%,\ 6) - \$8{,}000\,(A/F,\ 15\%,\ 6) = \$28{,}148.40.$$

This value (\$28,148.40) is the minimum annual dollar savings needed to justify the space-heating system. This equates to

$$\frac{\$28{,}148.40}{\$0.10/\text{kWh}} = 281{,}480 \text{ kWh per year.}$$

If the space-heating system can save 281,480 kWh per year, it is economically justified (exactly 15% is earned on the beginning-of-year unrecovered investment). Any savings greater than 281,480 kWh per year (such as the original estimate of 300,000 kWh per year) will serve to make this project even more attractive.

EXAMPLE 5-11

Avoid Getting Fleeced on an Auto Lease

Automobile leases are built around three factors: negotiated sales price, residual value, and interest rate. The residual value is what the dealership expects the car's value will be when the vehicle is returned at the end of the lease period. The monthly cost of the lease is the capital recovery amount determined by using these three factors.

(a) Determine the monthly lease payment for a car that has an agreed-upon sales price of \$34,995, an APR of 9% compounded monthly, and an estimated residual value of \$20,000 at the end of a 36-month lease.* An up-front payment of \$3,000 is due when the lease agreement (contract) is signed.
(b) If the estimated residual value is raised to \$25,000 by the dealership to get your business, how much will the monthly payment be?

Solution

(a) The effective sales price is \$31,995 (\$34,995 less the \$3,000 due at signing). The monthly interest rate is 9%/12 = 0.75% per month. So the capital recovery amount is:

$$\begin{aligned}\text{CR} &= \$31{,}995(A/P,\ 0.75\%,\ 36) - \$20{,}000(A/F,\ 0.75\%,\ 36)\\ &= \$1{,}017.44 - \$486\\ &= \$531.44 \text{ per month.}\end{aligned}$$

* There are many excellent Web sites offering free calculators for car loans, home mortgages, life insurance, and credit cards. In this regard, check out www.choosetosave.org.

(b) The capital recovery amount is now $1,017.44 − $25,000 $(A/F, 0.75\%, 36)$ = $409.94 per month. But the customer might experience an actual residual value of less than $25,000 and have to pay the difference in cash when the car is returned after 36 months. This is the "trap" that many experience when they lease a car, so be careful not to drive the car excessively or to damage it in any way.

5.6 The Internal Rate of Return Method

The IRR method is the most widely used rate-of-return method for performing engineering economic analyses. It is sometimes called by several other names, such as the *investor's method*, the *discounted cash-flow method*, and the *profitability index*.

This method solves for the interest rate that equates the equivalent worth of an alternative's cash inflows (receipts or savings) to the equivalent worth of cash outflows (expenditures, including investment costs). Equivalent worth may be computed using any of the three methods discussed earlier. The resultant interest rate is termed the *Internal Rate of Return (IRR)*. The *IRR* is sometimes referred to as the breakeven interest rate.

For a single alternative, from the lender's viewpoint, the IRR is not positive unless (1) both receipts and expenses are present in the cash-flow pattern, and (2) the sum of receipts exceeds the sum of all cash outflows. Be sure to check both of these conditions in order to avoid the unnecessary work involved in finding that the IRR is *negative*. (Visual inspection of the total net cash flow will determine whether the IRR is zero or less.)

Using a PW formulation, we see that the IRR is the $i'\%^*$ at which

$$\sum_{k=0}^{N} R_k(P/F, i'\%, k) = \sum_{k=0}^{N} E_k(P/F, i'\%, k), \tag{5-6}$$

where R_k = net revenues or savings for the kth year;
E_k = net expenditures, including any investment costs for the kth year;
N = project life (or study period).

Once i' has been calculated, it is compared with the MARR to assess whether the alternative in question is acceptable. If $i' \geq$ MARR, the alternative is acceptable; otherwise, it is not.

IRR Decision Rule: If IRR $\geq$ MARR, the project is economically justified.

A popular variation of Equation (5-6) for computing the IRR for an alternative is to determine the i' at which its *net* PW is zero. In equation form, the IRR is the

* i' is often used in place of i to mean the interest rate that is to be determined.

value of i' at which

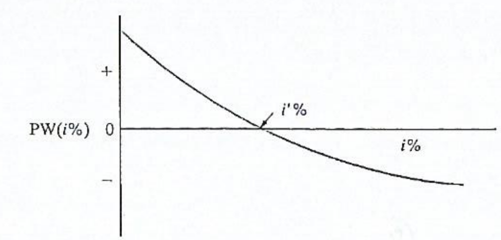
$$\text{PW} = \sum_{k=0}^{N} R_k(P/F, i'\%, k) - \sum_{k=0}^{N} E_k(P/F, i'\%, k) = 0. \tag{5-7}$$

For an alternative with a single investment cost at the present time followed by a series of positive cash inflows over N, a graph of PW versus the interest rate typically has the general convex form shown in Figure 5-3. The point at which PW = 0 in Figure 5-3 defines $i'\%$, which is the project's IRR. The value of $i'\%$ can also be determined as the interest rate at which FW = 0 or AW = 0.

Another way to interpret the IRR is through an *investment-balance diagram*. Figure 5-4 shows how much of the original investment in an alternative is still to be recovered as a function of time. The downward arrows in Figure 5-4 represent annual returns, $(R_k - E_k)$ for $1 \leq k \leq N$, against the unrecovered investment, and the dashed lines indicate the opportunity cost of interest, or profit, on the beginning-of-year investment balance. *The IRR is the value of i' in Figure 5-4 that causes the unrecovered investment balance to exactly equal zero at the end of the study period (year N)* and thus represents the *internal* earning rate of a project. It is important to notice that $i'\%$ is calculated on the *beginning-of-year unrecovered* investment through the life of a project rather than on the total initial investment.

The method of solving Equations (5-6) and (5-7) normally involves trial-and-error calculations until the $i'\%$ is converged upon or can be interpolated. Example 5-12 presents a typical solution. We also demonstrate how a spreadsheet application significantly assists in the computation of the IRR.

Figure 5-3 Plot of PW versus Interest Rate



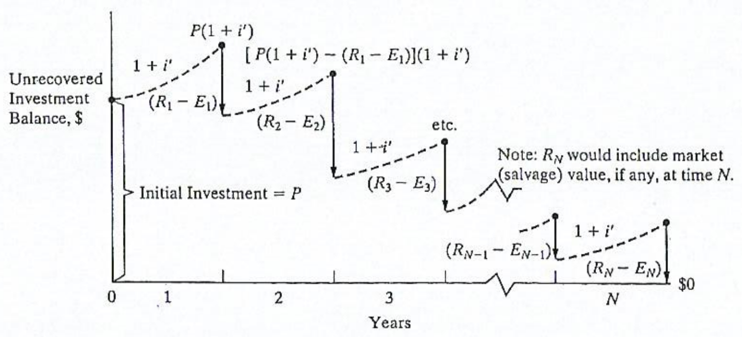


Figure 5-4 Investment-Balance Diagram Showing IRR

EXAMPLE 5-12

Economic Desirability of a Project Using the IRR Method

AMT, Inc., is considering the purchase of a digital camera for the maintenance of design specifications by feeding digital pictures directly into an engineering workstation where computer-aided design files can be superimposed over the digital pictures. Differences between the two images can be noted, and corrections, as appropriate, can then be made by design engineers. The capital investment requirement is $345,000 and the estimated market value of the system after a six-year study period is $115,000. Annual revenues attributable to the new camera system will be $120,000, whereas additional annual expenses will be $22,000. You have been asked by management to determine the IRR of this project and to make a recommendation. The corporation's MARR is 20% per year. Solve first by using linear interpolation and then by using a spreadsheet.

Solution by Linear Interpolation

In this example, we can easily see that the sum of positive cash flows ($835,000) exceeds the sum of negative cash flows ($455,000). Thus, it is likely that a positive-valued IRR can be determined. By writing an equation for the PW of the project's total net cash flow and setting it equal to zero, we can compute the IRR:

$$PW = 0 = -\$345{,}000 + (\$120{,}000 - \$22{,}000)(P/A, i'\%, 6)$$
$$+ \$115{,}000(P/F, i'\%, 6)$$
$$i'\% = ?$$

To use linear interpolation, we first need to try a few values for i'. A good starting point is to use the MARR.

$$\text{At } i' = 20\%: \quad PW = -\$345{,}000 + \$98{,}000(3.3255) + \$115{,}000(0.3349)$$
$$= +\$19{,}413$$

Since the PW is positive at 20%, we know that $i' > 20\%$.

$$\text{At } i' = 25\%: \quad PW = -\$345{,}000 + \$98{,}000(2.9514) + \$115{,}000(0.2621)$$
$$= -\$25{,}621$$

Now that we have both a positive and a negative PW, the answer is bracketed ($20\% \leq i'\% \leq 25\%$). The dashed curve in Figure 5-5 is what we are linearly approximating. The answer, $i'\%$, can be determined by using the similar triangles represented by dashed lines in Figure 5-5.

$$\frac{\text{line BA}}{\text{line BC}} = \frac{\text{line dA}}{\text{line de}}.$$

Here, BA is the line segment $B - A = 25\% - 20\%$. Thus,

$$\frac{25\% - 20\%}{\$19{,}413 - (-\$25{,}621)} = \frac{i'\% - 20\%}{\$19{,}413 - \$0}$$
$$i' \approx 22.16\%.$$

Because the IRR of the project (22.16%) is greater than the MARR, the project is acceptable.

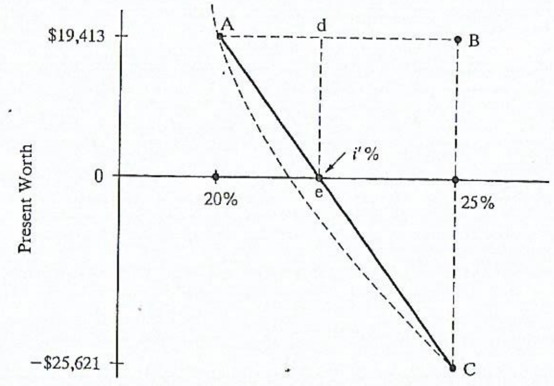


Figure 5-5 Use of Linear Interpolation to Find the Approximation of IRR for Example 5-12

	A	B		C	D	E	
1	MARR =		20%				
2	Capital investment =	$	345,000				
3	Market value =	$	115,000				
4	Annual Revenues =	$	120,000				
5	Annual Expenses =	$	22,000				
6	Study Period =		6 years				
7							
8	EOY	Cash Flow			EOY	Cash Flow	
9	0	$	(345,000)		0	$	(345,000)
10	1	$	98,000		1	$	98,000
11	2	$	98,000		2	$	98,000
12	3	$	98,000		3	$	98,000
13	4	$	98,000		4	$	98,000
14	5	$	98,000		5	$	98,000
15	6	$	98,000		6	$	213,000
16	6	$	115,000				
17							
18					Internal Rate of Return		22.03%

= −B2 (B9); = B$4−B$5 (B10); = B3 (B16); = B9 (E9); = B15 + B16 (E15); = IRR(E9:E15, B1) (E18)

Figure 5-6 Spreadsheet Solution, Example 5-12

Spreadsheet Solution

Figure 5-6 displays the spreadsheet solution for this example. The Excel function IRR(*range, guess*) requires the net cash flows for the study period and an initial guess for the IRR value (using the MARR is a good idea). Unlike the trial-and-error approach required when solving for the IRR by hand, the Excel IRR function is direct and simple. In cell E18, the IRR is calculated to be 22.03%.

EXAMPLE 5-13

Evaluation of New Equipment Purchase, Using the Internal Rate of Return Method (Example 5-1 Revisited)

A piece of new equipment has been proposed by engineers to increase the productivity of a certain manual welding operation. The investment cost is \$25,000, and the equipment will have a market (salvage) value of \$5,000 at the end of its expected life of five years. Increased productivity attributable to the equipment will amount to \$8,000 per year after extra operating costs have been subtracted from the value of the additional production. Use a spreadsheet to evaluate the IRR of the proposed equipment. Is the investment a good one? Recall that the MARR is 20% per year.

Spreadsheet Solution

The spreadsheet solution for this problem is shown in Figure 5-7. In column E of Figure 5-7(a), the individual EOY cash flows for year five (net annual savings

	A	B	C	D	E
1	MARR =	20%			
2	Capital Investment =	\$ 25,000			
3	Market Value =	\$ 5,000			
4	Useful Life =	5			
5	Net Annual Savings =	\$ 8,000			
6					
7	EOY	Cash Flow		EOY	Cash Flow
8	0	\$ (25,000)		0	\$ (25,000)
9	1	\$ 8,000		1	\$ 8,000
10	2	\$ 8,000		2	\$ 8,000
11	3	\$ 8,000		3	\$ 8,000
12	4	\$ 8,000		4	\$ 8,000
13	5	\$ 8,000		5	\$ 13,000
14	5	\$ 5,000			
15					
16				IRR =	21.58%

= −B2

= \$B\$5

= B3

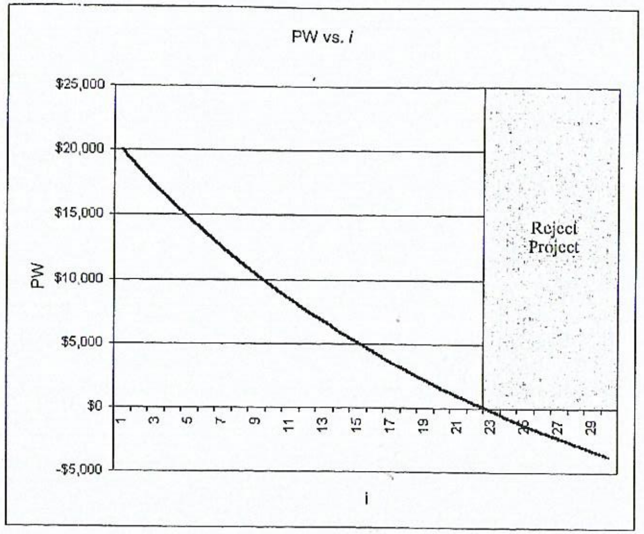
= B8

= B13 + B14

= IRR(E8:E13, B1)

(a) Direct Computation of IRR

Figure 5-7 Spreadsheet Solution, Example 5-13



(b) Graphical Determination of IRR

Figure 5-7 *Continued*

and market value) are combined into a single entry for the direct computation of the IRR via the IRR function. The IRR for the proposed piece of equipment is 21.58%, which is greater than the MARR of 20%. Thus, we conclude that the new equipment is economically feasible. We reached the same conclusion by using the equivalent worth method in previous examples, summarized as follows:

Example 5-1 PW(20%) = \$934.29;

Example 5-6 FW(20%) = \$2,324.80; and

Example 5-8 AW(20%) = \$312.40.

Figure 5-7(b) is a graph of the PW of the proposed equipment as a function of the interest rate, *i*. For the given problem data, the graph shows the IRR to be approximately 22%. This graph can be used to get a feel for how the IRR fluctuates with changes in the original cash-flow estimates. For example, if the net annual savings estimate [cell B5 of Figure 5-7(a)] is revised to be \$7,500, the graph would update and show the IRR to be 19%, and the project would no longer be economically viable.

A final point needs to be made for Example 5-13. The investment-balance diagram is provided in Figure 5-8, and the reader should notice that $i' = 21.577\%$ is a rate of return calculated on the beginning-of-year unrecovered investment. The IRR is *not* an average return each year based on the total investment of \$25,000.

5.6.1 Installment Financing

A rather common application of the IRR method is in so-called *installment financing* types of problems. These problems are associated with financing arrangements for purchasing merchandise "on time." The total interest, or finance, charge is often paid by the borrower on the basis of what amount is owed at the beginning of the loan *instead* of on the unpaid loan balance, as illustrated in Figure 5-8. Usually, the average unpaid loan balance is about one-half of the initial amount borrowed. Clearly, a finance charge based solely on the *entire* amount of money borrowed involves payment of interest on money not actually borrowed for the full term. This practice leads to an actual interest rate that often greatly exceeds the stated interest rate. To determine the true interest rate being charged in such cases, the IRR method is frequently employed. Examples 5-14, 5-15, and 5-16 are representative installment-financing problems.

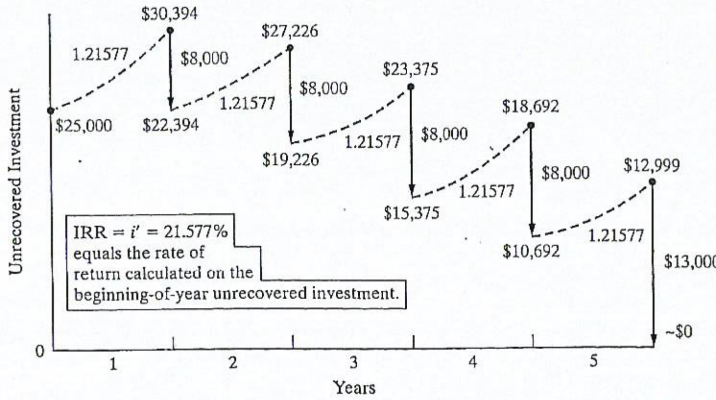


Figure 5-8 Investment-Balance Diagram for Example 5-13

EXAMPLE 5-14 **Do You Know What Your Effective Interest Rate Is?**

In 1915, Albert Epstein allegedly borrowed \$7,000 from a large New York bank on the condition that he would repay 7% of the loan every three months, until a total of 50 payments had been made. At the time of the 50th payment, the \$7,000 loan would be completely repaid. Albert computed his annual interest rate to be $[0.07(\$7{,}000) \times 4]/\$7{,}000 = 0.28$ (28%).

(a) What true *effective* annual interest rate did Albert pay?

(b) What, if anything, was wrong with his calculation?

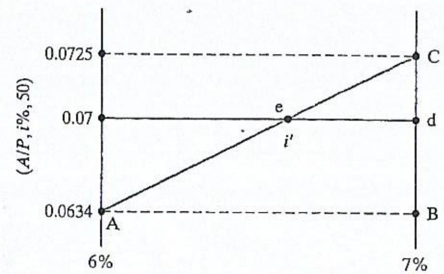
Solution

(a) The true interest rate per quarter is found by equating the equivalent value of the amount borrowed to the equivalent value of the amounts repaid. Equating the AW amounts per quarter, we find

$$\$7{,}000(A/P, i'\%/\text{quarter}, 50 \text{ quarters}) = 0.07(\$7{,}000) \text{ per quarter},$$
$$(A/P, i'\%, 50) = 0.07.$$

Linearly interpolating to find i'%/quarter by using similar triangles is the next step:

$$(A/P, 6\%, 50) = 0.0634,$$
$$(A/P, 7\%, 50) = 0.0725.$$



$$\frac{\text{line de}}{\text{line BA}} = \frac{\text{line Cd}}{\text{line CB}}$$

$$\frac{7\% - i'\%}{7\% - 6\%} = \frac{0.0725 - 0.07}{0.0725 - 0.0634},$$

$$i'\% = 7\% - 1\%\left(\frac{0.0025}{0.0091}\right),$$

or $i'\% \simeq 6.73\%$ per quarter.

Now we can compute the effective $i'\%$ per year that Albert was paying:

$$i'\% = [(1.0673)^4 - 1]100\%$$

$$\simeq 30\% \text{ per year.}$$

(b) Even though Albert's answer of 28% is close to the true value of 30%, his calculation is insensitive to how long his payments were made. For instance, he would get 28% for an answer when 20, 50, or 70 quarterly payments of $490 were made! For 20 quarterly payments, the true effective interest rate is 14.5% per year, and for 70 quarterly payments, it is 31% per year. As more payments are made, the true annual effective interest rate being charged by the bank will increase, but Albert's method would not reveal by how much.

Another Solution

When the initial loan amount, the payment amount, and the number of payments are known, Excel has a useful financial function, RATE (*nper, pmt, pv*), that will return the interest rate per period. For this example,

RATE(50, −490, 7000) = 6.73%.

This is the same quarterly interest rate we obtained via linear interpolation in Part (a).

EXAMPLE 5-15 Be Careful with "Fly-by-Night" Financing!

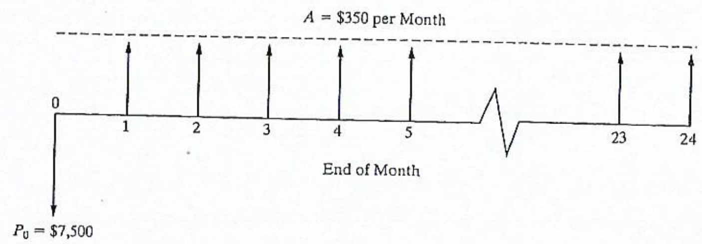
The Fly-by-Night finance company advertises a "bargain 6% plan" for financing the purchase of automobiles. To the amount of the loan being financed, 6% is added for each year money is owed. This total is then divided by the number of months over which the payments are to be made, and the result is the amount of the monthly payments. For example, a woman purchases a $10,000 automobile under this plan and makes an initial cash payment of $2,500. She wishes to pay the $7,500 balance in 24 monthly payments:

Purchase price	=	$10,000
−Initial payment	=	2,500
= Balance due, (P_0)	=	7,500
+6% finance charge = 0.06 × 2 years × $7,500	=	900
= Total to be paid	=	8,400
∴ Monthly payments (A) = $8,400/24	=	$350

What effective annual rate of interest does she actually pay?

Solution

Because there are to be 24 payments of $350 each, made at the end of each month, these constitute an annuity (A) at some unknown rate of interest, $i'\%$, that should be computed only upon the unpaid balance instead of on the entire $7,500 borrowed. A cash-flow diagram of this situation is shown below.



In this example, the amount owed on the automobile (i.e., the initial unpaid balance) is $7,500, so the following equivalence expression is employed to compute the unknown monthly interest rate:

$$P_0 = A(P/A, i'\%, N),$$

$$\$7{,}500 = \$350/\text{month}(P/A, i'\%, 24 \text{ months}),$$

$$(P/A, i'\%, 24) = \frac{\$7{,}500}{\$350} = 21.43.$$

Examining the interest tables for P/A factors at $N = 24$ that come closest to 21.43, we find that $(P/A, 3/4\%, 24) = 21.8891$ and $(P/A, 1\%, 24) = 21.2434$.

Using the linear interpolation procedure, the IRR is computed as 0.93% per month*, since payments are monthly. The nominal rate paid on the borrowed money is 0.93%(12) = 11.16% compounded monthly. This corresponds to an effective annual interest rate of $[(1 + 0.0093)^{12} - 1] \times 100\% \cong 12\%$. What appeared at first to be a real bargain turns out to involve effective annual interest at twice the stated rate. The reason is that, on the average, only $3,750 is borrowed over the two-year period, but interest on $7,500 over 24 months was charged by the finance company.

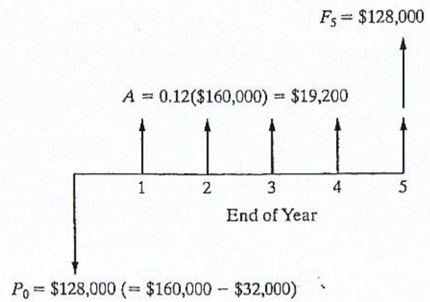
* Using Excel, RATE(24, −350, 7500) = 0.93%.

EXAMPLE 5-16 Effective Interest Rate for Purchase of New Aircraft Equipment

A small airline executive charter company needs to borrow $160,000 to purchase a prototype synthetic vision system for one of its business jets. The SVS is intended to improve the pilots' situational awareness when visibility is impaired. The local (and only) banker makes this statement: "We can loan you $160,000 at a very favorable rate of 12% per year for a five-year loan. However, to secure this loan, you must agree to establish a checking account (with no interest) in which the *minimum* average balance is \$32,000. In addition, your interest payments are due at the end of each year, and the principal will be repaid in a lump-sum amount at the end of year five." What is the true effective annual interest rate being charged?

Solution

The cash-flow diagram from the banker's viewpoint appears below. When solving for an unknown interest rate, it is good practice to draw a *cash-flow diagram* prior to writing an equivalence relationship. Notice that $P_0 =$ \$160,000 − \$32,000 = \$128,000. Because the bank is requiring the company to open an account worth \$32,000, the bank is only \$128,000 out of pocket. This same principal applies to F_5 in that the company only needs to repay \$128,000 since the \$32,000 on deposit can be used to repay the original principal.



The interest rate (IRR) that establishes equivalence between positive and negative cash flows can now easily be computed:*

$$P_0 = F_5(P/F, i'\%, 5) + A(P/A, i'\%, 5),$$

$$\$128{,}000 = \$128{,}000(P/F, i'\%, 5) + \$19{,}200(P/A, i'\%, 5).$$

If we try $i' = 15\%$, we discover that \$128,000 = \$128,000. Therefore, the true effective interest rate is 15% per year.

* Several Web sites provide excellent tutorials for equivalent worth (e.g., PW) and rate-of-return calculations. For example, see www.investopedia.com and www.datadynamica.com/IRR.asp.

5.6.2 Difficulties Associated with the IRR Method

The PW, AW, and FW methods assume that net receipts less expenses (positive recovered funds) each period are reinvested at the MARR during the study

period, N. However, the IRR method is not limited by this common assumption and measures the internal earning rate of an investment.*

Two difficulties with the IRR method are its computational difficulty and the occurrence of multiple IRRs in some types of problems. A procedure for dealing with seldom-experienced multiple rates of return is discussed and demonstrated in Appendix 5-A. Generally speaking, multiple rates are not meaningful for decision-making purposes, and another method of evaluation (e.g., PW) should be utilized.

Another possible drawback to the IRR method is that it must be carefully applied and interpreted in the analysis of two or more alternatives when only one of them is to be selected (i.e., mutually exclusive alternatives). This is discussed further in Chapter 6. The key advantage of the method is its widespread acceptance by industry, where various types of rates of return and ratios are routinely used in making project selections. The difference between a project's IRR and the required return (i.e., MARR) is viewed by management as a measure of investment safety. A large difference signals a wider margin of safety (or less relative risk).

5.7 The External Rate of Return Method†

The reinvestment assumption of the IRR method may not be valid in an engineering economy study. For instance, if a firm's MARR is 20% per year and the IRR for a project is 42.4%, it may not be possible for the firm to reinvest net cash proceeds from the project at much more than 20%. This situation, coupled with the computational demands and possible multiple interest rates associated with the IRR method, has given rise to other rate of return methods that can remedy some of these weaknesses.

One such method is the ERR method. It directly takes into account the interest rate (ϵ) external to a project at which net cash flows generated (or required) by the project over its life can be reinvested (or borrowed). If this external reinvestment rate, which is usually the firm's MARR, happens to equal the project's IRR, then the ERR method produces results identical to those of the IRR method.

In general, *three* steps are used in the calculating procedure. First, all net cash *outflows* are discounted to time zero (the present) at $\epsilon\%$ per compounding period. Second, all net cash *inflows* are compounded to period N at $\epsilon\%$. Third, the ERR, which is the interest rate that establishes equivalence between the two quantities, is determined. The *absolute value* of the present equivalent worth of the net cash outflows at $\epsilon\%$ (first step) is used in this last step. In equation form, the ERR is the $i'\%$ at which

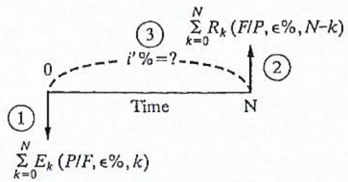
$$\sum_{k=0}^{N} E_k(P/F, \epsilon\%, k)(F/P, i'\%, N) = \sum_{k=0}^{N} R_k(F/P, \epsilon\%, N-k), \quad (5\text{-}8)$$

* See H. Bierman and S. Smidt, *The Capital Budgeting Decision: Economic Analysis of Investment Projects*, 8th ed. (Upper Saddle River, NJ: Prentice Hall, 1993). The term *internal* rate of return means that the value of this measure depends only on the cash flows from an investment and not on any assumptions about reinvestment rates: "One does not need to know the reinvestment rates to compute the IRR. However, one may need to know the reinvestment rates to compare alternatives" (p. 60).

† This method is also known as the "modified internal rate of return" (MIRR) method. For example, see C. S. Park and G. P. Sharp-Bette, *Advanced Engineering Economy*. (New York: John Wiley & Sons, 1990), 223–226.

where R_k = excess of receipts over expenses in period k;
E_k = excess of expenditures over receipts in period k;
N = project life or number of periods for the study;
ϵ = external reinvestment rate per period.

Graphically, we have the following (the numbers relate to the three steps):



A project is acceptable when $i'\%$ of the ERR method is greater than or equal to the firm's MARR.

ERR Decision Rule: If ERR ≥ MARR, the project is economically justified.

The ERR method has two basic advantages over the IRR method:

1. It can usually be solved for directly, without needing to resort to trial and error.
2. It is not subject to the possibility of multiple rates of return. (*Note:* The multiple-rate-of-return problem with the IRR method is discussed in Appendix 5-A.)

EXAMPLE 5-17 **Calculation of the ERR**

Referring to Example 5-13, suppose that ϵ = MARR = 20% per year. What is the project's ERR, and is the project acceptable?

Solution

By utilizing Equation (5-8), we have the following relationship to solve for i':

$$\$25{,}000(F/P, i'\%, 5) = \$8{,}000(F/A, 20\%, 5) + \$5{,}000,$$

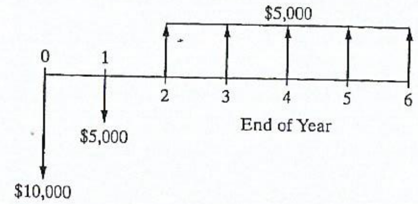
$$(F/P, i'\%, 5) = \frac{\$64{,}532.80}{\$25{,}000} = 2.5813 = (1+i')^5,$$

$$i' = 20.88\%.$$

Because $i' >$ MARR, the project is justified, but just barely.

EXAMPLE 5-18 **Determining the Acceptability of a Project, Using ERR**

When ϵ = 15% and MARR = 20% per year, determine whether the project (whose net cash-flow diagram appears next) is acceptable. Notice in this example that the use of an $\epsilon\%$ different from the MARR is illustrated. This might occur if, for some reason, part or all of the funds related to a project are "handled" outside the firm's normal capital structure.



Solution

$$E_0 = \$10{,}000 \ (k = 0),$$
$$E_1 = \$5{,}000 \ (k = 1),$$
$$R_k = \$5{,}000 \text{ for } k = 2, 3, \ldots, 6,$$

$$[\$10{,}000 + \$5{,}000(P/F, 15\%, 1)](F/P, i'\%, 6) = \$5{,}000(F/A, 15\%, 5);$$
$$i'\% = 15.3\%.$$

The $i'\%$ is less than the MARR = 20%; therefore, this project would be unacceptable according to the ERR method.

5.8 The Payback (Payout) Period Method

All methods presented thus far reflect the *profitability* of a proposed alternative for a study period of N. The payback method, which is often called the *simple payout method*, mainly indicates a project's *liquidity* rather than its profitability. Historically, the payback method has been used as a measure of a project's riskiness, since liquidity deals with how fast an investment can be recovered. A low-valued payback period is considered desirable. Quite simply, the payback method calculates the number of years required for cash inflows to just equal cash outflows. Hence, the simple payback period is the *smallest* value of $\theta (\theta \leq N)$ for which this relationship is satisfied under our normal EOY cash-flow convention. For a project where all capital investment occurs at time 0, we have

$$\sum_{k=1}^{\theta} (R_k - E_k) - I \geq 0. \quad (5\text{-}9)$$

The *simple* payback period, θ, ignores the time value of money and all cash flows that occur after θ. If this method is applied to the investment project in Example 5-13, the number of years required for the undiscounted sum of cash inflows to exceed the initial investment is four years. This calculation is shown in Column 3 of Table 5-2. Only when $\theta = N$ (the last time period in the planning horizon) is the market (salvage) value included in the determination of a payback period. As can be seen from Equation (5-9), the payback period does not indicate anything about project desirability except the speed with which the investment will be recovered. The payback period can produce misleading results, and it is recommended as supplemental information only in conjunction with one or more of the five methods previously discussed.

Sometimes, the *discounted* payback period, $\theta'(\theta' \leq N)$, is calculated so that the time value of money is considered. In this case,

$$\sum_{k=1}^{\theta'}(R_k - E_k)(P/F, i\%, k) - I \geq 0, \qquad (5\text{-}10)$$

where $i\%$ is the MARR, I is the capital investment usually made at the present time ($k = 0$), and θ' is the smallest value that satisfies Equation (5-10). Table 5-2 (Columns 4 and 5) also illustrates the determination of θ' for Example 5-13. Notice that θ' is the first year in which the cumulative discounted cash inflows exceed the \$25,000 capital investment. Payback periods of three years or less are often desired in U.S. industry, so the project in Example 5-13 could be *rejected*, even though it is profitable. [IRR = 21.58%, PW(20%) = \$934.29.] The simple and discounted payback periods are shown graphically in Figure 5-9.

Calculation of the Simple Payback Period (θ) and the Discounted Payback Period (θ') at MARR = 20% for Example 5-13[a]

Column 1 End of Year k	Column 2 Net Cash Flow	Column 3 Cumulative PW at $i = 0\%$/yr through Year k	Column 4 PW of Cash Flow at $i = 20\%$/yr	Column 5 Cumulative PW at $i = 20\%$/yr through Year k
0	−\$25,000	−\$25,000	−\$25,000	−\$25,000
1	8,000	−17,000	6,667	−18,333
2	8,000	−9,000	5,556	−12,777
3	8,000	−1,000	4,630	−8,147
4	8,000	+7,000	3,858	−4,289
5	13,000		5,223	+934

↑ $\theta = 4$ years because the cumulative balance turns positive at EOY 4.

↑ $\theta' = 5$ years because the cumulative discounted balance turns positive at EOY 5.

[a] Notice that $\theta' \geq \theta$ for MARR $\geq 0\%$.

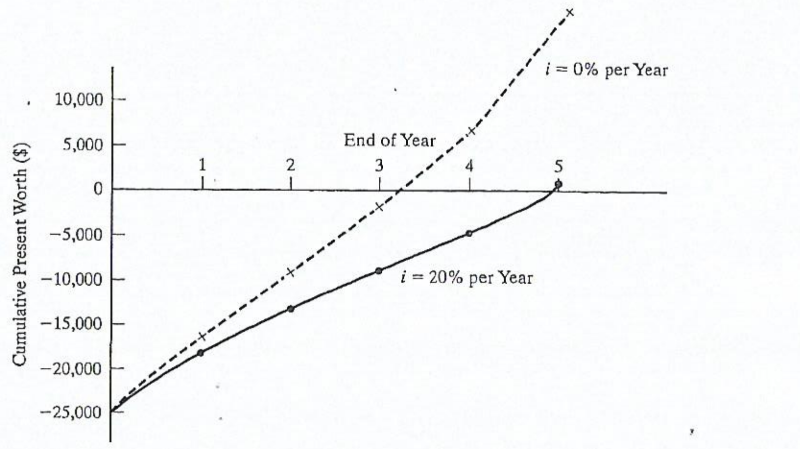


Figure 5-9 Graph of Cumulative PW for Example 5-13

This variation (θ') of the simple payback period produces the *breakeven life* of a project, in view of the time value of money. However, neither payback period calculation includes cash flows occurring after θ (or θ'). This means that θ (or θ') may not take into consideration the entire useful life of physical assets. Thus, these methods will be misleading if one alternative has a longer (less desirable) payback period than another but produces a higher rate of return (or PW) on the invested capital.

Using the payback period to make investment decisions should generally be avoided except as a secondary measure of how quickly invested capital will be recovered, which is an indicator of project risk. The simple payback and discounted payback period methods tell us how long it takes cash inflows from a project to accumulate to equal (or exceed) the project's cash outflows. The longer it takes to recover invested monies, the greater is the perceived riskiness of a project.

5.9 CASE STUDY—A Proposed Capital Investment to Improve Process Yield

Many engineering projects aim at improving facility utilization and process yields. This case study illustrates an engineering economy analysis related to the redesign of a major component in the manufacture of semiconductors.

Semiconductor manufacturing involves taking a flat disc of silicon, called a wafer, and depositing many layers of material on top of it. Each layer has a pattern on it that, upon completion, defines the electrical circuits of the finished microprocessor. Each 8-inch wafer has up to 100 microprocessors on it. However, the typical average yield of the production line is 75% good microprocessors per wafer.

At one local company, the process engineers responsible for the chemical-vapor-deposition (CVD) tool (i.e., process equipment) that deposits *one* of the many layers have an idea for improving overall yield. They propose to improve this tool's vacuum with a redesign of one of its major components. The engineers believe the project will result in a 2% increase in the average production yield of nondefective microprocessors per wafer.

This company has only one CVD tool, and it can process 10 wafers per hour. The process engineers have determined that the CVD tool has an average utilization rate (i.e., "time running") of 80%. A wafer costs $5,000 to manufacture, and a good microprocessor can be sold for $100. These semiconductor fabrication plants ("fabs") operate 168 hours per week, and all good microprocessors produced can be sold.

The capital investment required for the project is $250,000, and maintenance and support expenses are expected to be $25,000 per month. The lifetime of the modified tool will be five years, and the company uses a 12% MARR per year (compounded monthly) as its "hurdle rate."

Before implementing the proposed engineering solution, top management has posed the following questions to you (hired as an independent consultant) to evaluate the merits of the proposal:

(a) Based on the PW method, should the project be approved?

(b) If the achievable improvement in production yield has been overestimated by the process engineers, at what percent yield improvement would the project breakeven?

Solution

You start your economic evaluation of the engineering proposal by first calculating the production rate of wafers. The average number of wafers per week is

$$(10 \text{ wafers/hour}) \times (168 \text{ hours/week}) \times (0.80) = 1{,}344.$$

Since the cost per wafer is $5,000 and good microprocessors can be sold for $100 each, you determine that profit is earned on each microprocessor produced and sold after the 50th microprocessor on each wafer. Thus, the 2% increase in production yield is all profit (i.e., from 75 good microprocessors per wafer on the average to 76.5). The corresponding additional profit per wafer is $150. The added profit per month, assuming a month is (52 weeks/year ÷ 12 months per year) = 4.333 weeks, is

$$(1{,}344 \text{ wafers/week})(4.333 \text{ weeks/month})(\$150/\text{wafer}) = \$873{,}533.$$

Therefore, the PW of the project is

$$\begin{aligned} \text{PW}(1\%) &= -\$250{,}000 - \$25{,}000(P/A, 1\% \text{ per month}, 60 \text{ months}) \\ &\quad + \$873{,}533(P/A, 1\%, 60) \\ &= \$37{,}898{,}813. \end{aligned}$$

You advise company management that, based on PW, the project *should* be undertaken.

It is known that at the breakeven point, profit equals zero. That is, the PW of the project is equal to zero, or PW of costs = PW of revenues. In other words,

$$\$1{,}373{,}875 = (1{,}344 \text{ wafers/week}) \times (4.333 \text{ weeks/month}) \times (\$X/\text{wafer}) \times (P/A, 1\%, 60),$$

where X = $100 times the number of extra microprocessors per wafer. Then,

$$\frac{\$1{,}373{,}875}{(1{,}344)(4.333)(44.955)} = X, \text{ or } X \cong \$5.25 \text{ per wafer.}$$

Thus, ($5.25/$100) = 0.0525 *extra* microprocessors per wafer (total of 75.0525) equates PW of costs to PW of revenues. This corresponds to a breakeven increase in yield of

$$\frac{1.5 \text{ die per wafer}}{0.0525 \text{ die per wafer}} = \frac{2.0\% \text{ increase}}{\text{breakeven increase}},$$

or breakeven increase in yield = 0.07%.

You advise management that an increase of only 0.07% in process yield would enable the project to breakeven. Thus, although management may believe that the process engineers have overestimated projected process yield improvements in the past, there is quite a bit of "economic safety margin" provided by the engineers in their current projection of process yield improvement as long as the other assumptions concerning the average utilization rate of the CVD tool, the wafer production rate, and the plant operating hours are valid.

5.10 Summary

Throughout this chapter, we have examined five basic methods for evaluating the financial *profitability* of a single project: PW, AW, FW, IRR, and ERR. These methods lead to the use of simple decision rules for economic evaluation of projects as presented in Table 5-3. Two supplemental methods for assessing a project's *liquidity* were also presented: the simple payback period and the discounted payback period. Computational procedures, assumptions, and acceptance criteria for all methods were discussed and illustrated with examples. Appendix B provides a listing of new abbreviations and notations that have been introduced in this chapter.

TABLE 5-3 Summary of Decision Rules

Economic Measure of Merit	Notes	Decision Rule
AW, PW, FW	All are functions of the MARR	If PW (or AW, FW) ≥ 0, accept the project; otherwise, reject it
IRR, ERR	Solve for unknown interest rate, i'	If $i' \geq$ MARR, accept the project; otherwise, reject it

Problems

Unless stated otherwise, discrete compounding of interest and end-of-period cash flows should be assumed in all problem exercises for the remainder of the book. All MARRs are "per year." The number in parentheses that follows each problem refers to the section from which the problem is taken.

5-1. "The higher the MARR, the higher the price that a company should be willing to pay for equipment that reduces annual operating expenses." Do you agree with this statement? Explain your answer. (5.2)

5-2. A project your firm is considering for implementation has these estimated costs and revenues: an investment cost of $50,000; maintenance costs that start at $5,000 at end-of-year (EOY) one and increase by $1,000 for each of the next four years, and then remain constant for the following five years; savings of $20,000 per year (EOY 1–10); and finally a resale value of $35,000 at EOY 10. If the project has a 10-year life and the firm's MARR is 10% per year, what is the present worth of the project? Is it a sound investment opportunity? (5.3)

5-3. Josh Ritchey has just been hired as a cost engineer by a large airlines company. Josh's first idea is to quit giving complimentary cocktails, wine, and beer to the international flying public. He calculates this will save 5,000,000 drinks per year, and each drink costs $0.50, for a total of $2.5 million per year. Instead of complimentary drinks, Josh estimates that the airlines company can sell 2,000,000 drinks at $5.00 per drink. The net savings would amount to $12.5 million per year! Josh's boss really likes the idea and agrees to give Josh a lump-sum bonus now equaling 0.1% of the present equivalent worth of three years of net savings. If the company's MARR is 20% per year, what is Josh's bonus? (5.3)

5-4. Evaluate a combined cycle power plant on the basis of the PW method when the MARR is 12% per year. Pertinent cost data are as follows: (5.3)

	Power Plant (thousands of $)
Investment cost	$13,000
Useful life	15 years
Market value (EOY 15)	$3,000
Annual operating expenses	$1,000
Overhaul cost—end of 5th year	$200
Overhaul cost—end of 10th year	$550

5-5. A new air conditioning system is to be installed in an office building. Purchasing and installing the system costs $100,000, and the company expects to save $30,000 every year on electricity bills. The company's MARR is 15% per year, maintenance costs are worked out to be $5,000 per year and the system's market value will be $20,000 at the end of 15 years. Use the PW method to determine whether the system should be installed. (5.1)

5-6. A large induced-draft fan is needed for an upgraded industrial process. The motor to drive this fan is rated at 100 horsepower, and the motor will operate at full load for 8,760 hours per year. The motor's efficiency is 90%. Because the motor is fairly large, a demand charge of $90 per kilowatt per year will be incurred in addition to an energy charge of $0.08 per kilowatt-hour. If the installed cost of the motor is $3,500, what is the present worth of the motor over a 10-year period when the MARR is 15% per year? (5.3)

5-7. A new municipal refuse-collection truck can be purchased for $84,000. Its expected useful life is six years, at which time its market value will be zero. Annual receipts less expenses will be approximately $18,000 per year over the six-year study period. Use the PW method and a MARR of 18% to determine whether this is a good investment. (5.3)

5-8. The winner of a state lottery will receive $5,000 per week for the rest of her life. If the winner's interest rate is 6.5% per year compounded weekly, what is the present worth of this jackpot? (5.3)

5-9. A new six-speed automatic transmission for automobiles offers an estimated 4% improvement in fuel economy compared to traditional four-speed transmissions in front-wheel drive cars. If a four-speed transmission car averages 30 miles per gallon and gasoline costs $3.00 per gallon, how much *extra* can a motorist pay for a fuel-efficient six-speed transmission? Assume that the car will be driven for 120,000 miles over its lifetime of 10 years. The motorist can earn 6% per year on investments. (5.3)

5-10. What is the maximum price you will pay for a bond with a face value of $1,000 and a coupon rate of 14%, paid annually, if you want a yield to maturity of 10%? Assume that the bond will mature in 10 years and the first payment will be received in one year. (5.3)

5-11. Last month Jim purchased $10,000 of U.S. Treasury bonds (their face value was $10,000). These bonds have a 30-year maturity period, and they pay 1.5% interest every three months (i.e., the APR is 6%, and Jim receives a check for $150 every three months). But interest rates for similar securities have since risen to a 7% APR because of interest rate increases by the Federal Reserve Board. In view of the interest-rate increase to 7%, what is the current value of Jim's bonds? (5.3)

5-12. A student buys a laptop for $1,500 which he hopes to resell upon completion of his degree. The laptop has an estimated salvage value of $300 at the end of 3 years. For an interest rate of 15%, find the equivalent annual cost of this investment using capital recovery with return. (5.5)

5-13. A company sold a $1,000,000 issue of bonds with a 15-year life, paying 4% interest per year. The bonds were sold at par value. If the company paid a selling fee of $50,000 and has an annual expense of $70,256 for mailing and record keeping, what is the true rate of interest that the company is paying for the borrowed money? (5.3)

5-14. A U.S. government bond matures in 10 years. Its quoted price is now 97.8, which means the buyer will pay $97.80 per $100 of the bond's face value. The bond pays 5% interest on its face value each year. If $10,000 (the face value) worth of these bonds is purchased now, what is the yield to the investor who holds the bonds for 10 years? (5.3)

5-15. The Western Railway Company (WRC) has been offered a 100-year contract to haul a fixed amount of coal each year from Wyoming to Illinois. Under the terms of the agreement, WRC will receive $4,200,000 now in exchange for its hauling services valued at $360,000 at the end of year (EOY) one, $375,000 at EOY 2 and continuing to grow by $15,000 per year through EOY 100. If WRC's cost of capital is 12% per year, is this a profitable agreement for WRC? (5.3)

5-16.

a. What is the CW, when $i = 10\%$ per year, of $1,500 per year, starting in year one and continuing forever; and $10,000 in year five, repeating every four years thereafter, and continuing forever? (5.3)

b. When $i = 10\%$ per year in this type of problem, what value of N, practically speaking, defines "forever"? (5.3)

5-17. A city is spending $20 million on a new sewage system. The expected life of the system is 40 years, and it will have no market value at the end of its life. Operating and maintenance expenses for the system are projected to average $0.6 million per year. If the city's MARR is 8% per year, what is the capitalized worth of the system? (5.3)

5-18. A foundation was endowed with $10,000,000 in July 2000. In July 2004, $3,000,000 was expended for facilities, and it was decided to provide $250,000 at the end of each year forever to cover operating expenses. The first operating expense was in July 2005, and the first replacement expense is in July 2009. If all money earns interest at 5% after the time of endowment, what amount would be available for capital replacements at the end of every fifth year forever? (*Hint:* Draw a cash-flow diagram first.) (5.3)

5-19. Vidhi is investing in some rental property in Collegeville and is investigating her income from the investment. She knows the rental revenue will increase each year, but so will the maintenance expenses. She has been able to generate the data that follow regarding this investment opportunity. Assume that all cash flows occur at the end of each year and that the purchase and sale of this property are not relevant to the study. If Vidhi's MARR = 6% per year, what is the FW of Vidhi's projected net income? (5.4)

Year	Revenue	Year	Expenses
1	$6,000	1	$3,100
2	6,200	2	3,300
3	6,300	3	3,500
4	6,400	4	3,700
5	6,500	5	3,900
6	6,600	6	6,100
7	6,700	7	4,300
8	6,800	8	4,500
9	6,900	9	4,700
10	7,000	10	4,900

5-20. In Problem 5-5, the projected savings in electricity bills rendered by the air conditioning system, which cost $100,000 to purchase and install, were calculated to be $30,000 per year. The system's market value after 15 years was estimated to be $20,000. Use the FW method to determine whether the purchase of this equipment would be justified if the air conditioning system's market value after 15 years would be zero. The company is currently using a MARR of 20%. (5.6)

5-21. Determine the FW of the following engineering project when the MARR is 15% per year. Is the project acceptable? **(5.4)**

	Proposal *A*
Investment cost	$10,000
Expected life	5 years
Market (salvage) value[a]	−$1,000
Annual receipts	$8,000
Annual expenses	$4,000

[a] A negative market value means that there is a net cost to dispose of an asset.

5-22. What are the PW and FW of a 20-year geometric cash-flow progression increasing at 2% per year if the first year amount is $1,020 and the interest rate is 10% per year? **(5.4)**

5-23. Fill in Table P5-23 below when P = $10,000, S = $2,000 (at the end of four years), and i = 15% per year. Complete the accompanying table and show that the equivalent uniform CR amount equals $3,102.12. **(5.5)**

5-24. Given that the purchase price of a machine is $1,000 and its market value at EOY four is $300, complete Table P5-24 below [values (a) through (f)], using an opportunity cost of 5% per year. Compute the equivalent uniform CR amount, using information from the completed table. **(5.5)**

5-25. An asset with an expected life of four years is to be purchased for $6,000. Annual revenue is estimated at $2,000, with increasing annual expenses of $400 at retirement. At the end of the fourth year, the asset is expected to be sold for $900, while $800 will be paid as transportation costs. The interest rate is 8% compounded annually. By using the AW method, determine whether the asset should be purchased or not. **(5.8)**

5-26. An office supply company has purchased a light-duty delivery truck for $15,000. It is anticipated that the purchase of the truck will increase the company's revenue by $10,000 annually, while the associated operating expenses are expected to be $3,000 per year. The truck's market value is expected to decrease by $2,500 each year it is in service. If the company plans to keep the truck for only two years, what is the annual worth of this investment? The MARR = 18% per year. **(5.5)**

Table for Problem 5-23

Year	Investment Beginning of Year	Opportunity Cost of Interest (i = 15%)	Loss in Value of Asset During Year	Capital Recovery Amount for Year
1	$10,000		$3,000	
2			$2,000	
3			$2,000	
4				

Table for Problem 5-24

Year	Investment at Beginning of Year	Opportunity Cost (5% per Year)	Loss in Value of Asset During Year	Capital Recovery Amount for Year
1	$1,000	$50	$(a)	$250
2	(b)	(c)	200	240
3	600	30	200	230
4	(d)	20	(e)	(f)

5-27. A company is considering constructing a plant to manufacture a proposed new product. The land costs $300,000, the building costs $600,000, the equipment costs $250,000, and $100,000 additional working capital is required. It is expected that the product will result in sales of $750,000 per year for 10 years, at which time the land can be sold for $400,000, the building for $350,000, and the equipment for $50,000. All of the working capital would be recovered at the EOY 10. The annual expenses for labor, materials, and all other items are estimated to total $475,000. If the company requires a MARR of 15% per year on projects of comparable risk, determine if it should invest in the new product line. Use the AW method. **(5.5)**

5-28. A 50-kilowatt gas turbine has an investment cost of $40,000. It costs another $14,000 for shipping, insurance, site preparation, fuel lines, and fuel storage tanks. The operation and maintenance expense for this turbine is $450 per year. Additionally, the hourly fuel expense for running the turbine is $7.50 per hour, and the turbine is expected to operate 3,000 hours each year. The cost of dismantling and disposing of the turbine at the end of its 8-year life is $8,000. **(5.5)**

a. If the MARR is 15% per year, what is the annual equivalent life-cycle cost of the gas turbine?

b. What percent of annual life-cycle cost is related to fuel?

5-29. Anderson County has 35 older-model school buses that will be salvaged for $5,000 each. These buses cost $144,000 per year for fuel and maintenance. Now the county will purchase 35 new school buses for $40,000 each. They will travel an average of 2,000 miles per day for a total of 360,000 miles per year. These new buses will save $10,000 per year in fuel (compared with the older buses) for the entire group of 35 buses. If the new buses will be driven for 15 years and the county's MARR is 6% per year, what is the equivalent uniform annual cost of the new buses if they have negligible market value after 15 years? **(5.5)**

5-30. Your sister just bought a new car having a sticker price (manufacturer's suggested retail price) of $36,000. She was crafty and was able to negotiate a sales price of $33,500 from the auto dealership. In addition, she received $4,500 for her old trade-in car under the U.S. government's "Cash for Clunkers" program. If her new car will have a resale value of $3,500 after seven years when your sister will shop for a replacement car, what is the annual capital recovery cost of your sister's purchase? The relevant interest rate is 8% per year, and your sister can afford to spend a maximum of $5,000 per year to finance the car (operating and other costs are extra). **(5.5)**

5-31. An environmentally friendly green home (99% air tight) costs about 8% more to construct than a conventional home. Most green homes can save 15% per year on energy expenses to heat and cool the dwelling. For a $250,000 conventional home, how much would have to be saved in energy expenses per year when the life of the home is 30 years and the interest rate is 10% per year? Assume the additional cost of a green home has no value at the end of 30 years. **(5.5)**

5-32. Your company is considering the introduction of a new product line. The initial investment required for this project is $500,000, and annual maintenance costs are anticipated to be $35,000. Annual operating costs will be directly proportional to the level of production at $7.50 per unit, and each unit of product can be sold for $50. If the MARR is 15% and the project has a life of five years, what is the minimum annual production level for which this project is economically viable? **(5.5)**

5-33. Stan Moneymaker has been informed of a major automobile manufacturer's plan to conserve on gasoline consumption through improved engine design. The idea is called "engine displacement," and it works by switching from 8-cylinder operation to 4-cylinder operation at approximately 40 miles per hour. Engine displacement allows enough power to accelerate from a standstill and to climb hills while also permitting the automobile to cruise at speeds over 40 miles per hour with little loss in driving performance.

The trade literature studied by Stan makes the claim that the engine displacement option will cost the customer an extra $1,200 on the automobile's sticker price. This option is expected to save 4 miles per gallon (an average of in-town and highway driving). A regular 8-cylinder engine in the car that Stan is interested in buying gets an average of 20 miles per gallon of gasoline. If Stan drives approximately 1,200 miles per month, how many months of ownership will be required to make this $1,200 investment pay for itself? Stan's opportunity cost of capital (i) is 0.5% per month, and gasoline costs $2.75 per gallon. **(5.3)**

5-34. The world's largest carpet maker has just completed a feasibility study of what to do with the 16,000 tons of overruns, rejects, and remnants it produces every year. The company's CEO launched the feasibility study by asking, why pay someone to dig coal out of the ground and then pay someone else to put our waste into a landfill? Why not just burn our own waste? The company is proposing to build a $10-million power plant to burn its waste as fuel, thereby saving $2.8 million a year in coal purchases. Company engineers have determined that the waste-burning plant will be environmentally sound, and after its four-year study period the plant can be sold to a local electric utility for $5 million. (5.6)

a. What is the IRR of this proposed power plant?

b. If the firm's MARR is 15% per year, should this project be undertaken?

5-35. Each year in the United States nearly 5 billion pounds of discarded carpet end up in land fills. In response to this situation, the carpet manufacturer in Problem 5-34 has decided to start a take-back program whereby obsolete carpet is reclaimed at the end of its useful life (about 7 years). The remanufactured carpet will then be recycled into the company's supply chain. The out-of-pocket (variable) cost of recycling is $1.00 per square yard of carpet, and the remanufactured carpet can be sold for $3.00 per square yard. If the recycling equipment costs $1 million and has no market value at the end of its eight-year life, how much carpet must be remanufactured annually to make this a profitable undertaking? The company's MARR is 15% per year. (5.5)

5-36. The doctors at a local hospital have requested an X-ray machine whose purchase will aid diagnostics and boost productivity. The investment cost will be $50,000 and the equipment's market value at the end of its expected life of 20 years will be $7,000. Increased productivity as a result of the purchase is calculated to be $10,000 per year, after extra operations costs have been subtracted from the value of the additional production. Evaluate the IRR of the machine using a 20% MARR. (5.12)

5-37. Manuel has the following investment planned: he is going to make a $12,000 deposit every year for the next 5 years, which means he plans to make the first deposit one year from now. One year after the last deposit has been made Manuel anticipates that he will need to make continuous withdrawals of $2,000 for the next 15 years. Calculate to the nearest percent the effective annual IRR Manuel is earning on his investment. (5.16)

5-38. A loan of $3,000 for a new, high-end laptop computer is to be repaid in 15 end-of-month payments (starting one month from now). The monthly payments are determined as follows.

Loan principal	$3,000
Interest for 15 months at 1.5% per month	675
Loan application fee	150
Total	$3,825

Monthly payment = $3,825/15 = $255

What nominal and effective interest rates per year are actually being paid? Hint: Draw a cash-flow diagram from the perspective of the lender. (5.6)

5-39. A person deposits $12,000 per year for 5 years, with the first deposit made one year from the present. One year after the last deposit, the person makes continuous withdrawals of $2,000 for the next 15 years. Find the effective annual IRR being earned on this investment to the nearest percent. (5.6)

5-40. In July of 2012, Taylor purchased 2,000 shares of XYZ common stock for $75,000. He then sold 1,000 shares of XYZ in July of 2013 for $39 per share. The remaining 1,000 shares were finally sold for $50 per share in July 2014. (5.6, 5.7)

a. Draw a cash-flow diagram of this situation.

b. What was Taylor's internal rate of return (IRR) on this investment?

c. What was the ERR on this investment if the external reinvestment rate is 8% per year?

5-41. Nowadays it is very important to reduce one's carbon "footprint" (how much carbon we produce in our daily lifestyles). Minimizing the use of fossil fuels and instead resorting to renewable sources of energy (e.g., solar energy) are vital to a "sustainable" lifestyle and a lower carbon footprint. Let's consider solar panels that prewarm the water fed to a conventional home water heater. The solar panels have an installed cost of $3,000, and they reduce the homeowner's energy bill by $30 per month. The residual value of the solar panels is negligible at the end of their 10-year life. What is the annual effective IRR of this investment? (5.6)

5-42. Sergeant Jess Frugal has the problem of running out of money near the end of each month (he gets paid once a month). Near his army base there is a payday lender company, called Predatory Lenders, Inc., that will give Jess a cash advance of $350 if he will repay the loan a month later with a post-dated check for $375. Almost as soon as Frugal's check for $375 clears the bank, he unfortunately must again borrow $350 to make ends meet. Jess's wife has gotten a bit concerned that her husband might be paying an exorbitant interest rate to this payday lender. Assuming Jess has repeated this borrowing and repayment scheme for 12 months in a row, what effective annual interest rate is he really paying? Is Jess's wife correct in her worry? Hint: Draw a cash-flow diagram from the viewpoint of the lender. (5.6)

5-43. A firm requires a crane which costs $80,000 and has an estimated salvage value of $10,000 at the end of a 6-year operating life. If the firm uses the rate of interest as 20% for the project evaluation, how much must be earned on capital recovery basis so that the firm recovers its invested capital plus earns a return on the capital committed to the equipment during its lifetime? (5.5)

5-44. To purchase a used automobile, you borrow $8,000 from Loan Shark Enterprises. They tell you the interest rate being charged is 1% per month for 35 months. They also charge you $200 for a credit investigation, so you leave with $7,800 in your pocket. The monthly payment they calculated for you is

$$\frac{\$8{,}000(0.01)(35) + \$8{,}000}{35} = \$308.57/\text{month}.$$

If you agree to these terms and sign their contract, what is the actual APR (annual percentage rate) that you are paying? (5.6)

5-45. Your boss has just presented you with the summary in the accompanying table of projected costs and annual receipts for a new product line. He asks you to calculate the IRR for this investment opportunity. What would you present to your boss, and how would you explain the results of your analysis? (It is widely known that the boss likes to see graphs of PW versus interest rate for this type of problem.) The company's MARR is 10% per year. (5.6)

End of Year	Net Cash Flow
0	−$450,000
1	−42,500
2	+92,800
3	+386,000
4	+614,600
5	−$202,200

5-46. A small company purchased now for $23,000 will lose $1,200 each year the first four years. An additional $8,000 invested in the company during the fourth year will result in a profit of $5,500 each year from the fifth year through the fifteenth year. At the end of 15 years, the company can be sold for $33,000.

a. Determine the IRR. (5.6)

b. Calculate the FW if MARR = 12%. (5.4)

c. Calculate the ERR when ϵ = 12%. (5.7)

5-47. A company has the opportunity to take over a redevelopment project in an industrial area of a city. No immediate investment is required, but it must raze the existing buildings over a four-year period and, at the end of the fourth year, invest $2,400,000 for new construction. It will collect all revenues and pay all costs for a period of 10 years, at which time the entire project, and properties thereon, will revert to the city. The net cash flows are estimated to be as follows:

Year End	Net Cash Flow
1	$500,000
2	300,000
3	100,000
4	−2,400,000
5	150,000
6	200,000
7	250,000
8	300,000
9	350,000
10	400,000

Tabulate the PW versus the interest rate and determine whether multiple IRRs exist. If so, use the ERR method when ϵ = 8% per year to determine a rate of return. (5.7)

5-48. The prospective exploration for oil in the outer continental shelf by a small, independent drilling company has produced a rather curious pattern of cash flows, as follows:

End of Year	Net Cash Flow
0	−$520,000
1–10	+200,000
10	−1,500,000

The $1,500,000 expense at EOY 10 will be incurred by the company in dismantling the drilling rig.

a. Over the 10-year period, plot PW versus the interest rate (i) in an attempt to discover whether multiple rates of return exist. (5.6)

b. Based on the projected net cash flows and results in Part (a), what would you recommend regarding the pursuit of this project? Customarily, the company expects to earn at least 20% per year on invested capital before taxes. Use the ERR method ($\epsilon = 20\%$). (5.7)

5-49. In this problem, we consider replacing an existing electrical water heater with an array of solar panels. The net installed investment cost of the panels is $1,400 ($2,000 less a 30% tax credit from the government). Based on an energy audit, the existing water heater uses 200 kilowatt hours (kWh) of electricity per month, so at $0.12 per kWh, the cost of operating the water heater is $24 per month. Assuming the solar panels can save the entire cost of heating water with electricity, answer the following questions. (5.6, 5.8)

a. What is the simple payback period for the solar panels?

b. What is the IRR of this investment if the solar panels have a life of 10 years?

5-50. A lathe costs $10,000 and will incur $200 per year in fixed costs and $25 per hour in variable costs. The lathe will be operational for 1,200 hours per year for 5 years and then sold for $2,000. The MARR is 15% per year. (5.9)

a. Determine the capital recovery cost of the lathe.

b. What is the EUAC of the lathe?

5-51. A computer call center is going to replace all of its incandescent lamps with more energy-efficient fluorescent lighting fixtures. The total energy savings are estimated to be $1,875 per year, and the cost of purchasing and installing the fluorescent fixtures is $4,900. The study period is five years, and terminal market values for the fixtures are negligible. (5.8)

a. What is the IRR of this investment?

b. What is the simple payback period of the investment?

c. Is there a conflict in the answers to Parts (a) and (b)? List your assumptions.

d. The simple payback "rate of return" is $1/\theta$. How close does this metric come to matching your answer in Part (a)?

5-52. A plasma arc furnace has an internal combustion temperature of 7,000°C and is being considered for the incineration of medical wastes at a local hospital. The initial investment is $300,000 and annual revenues are expected to be $175,000 over the six-year life of the furnace. Annual expenses will be $100,000 at the end of year one and will increase by $5,000 each year thereafter. The resale value of the furnace after six years is $20,000. (5.6, 5.8)

a. What is the simple payback period of the furnace?

b. What is the internal rate of return on the furnace?

5-53. Advanced Modular Technology (AMT) typically exhibits net annual revenues that increase over a fairly long period. In the long run, an AMT project may be profitable as measured by IRR, but its simple payback period may be unacceptable. Evaluate this AMT project using the IRR method when the company MARR is 15% per year and its maximum allowable payback period is three years. What is your recommendation? (5.6, 5.8)

Capital investment at time 0	$100,000
Net revenues in year k	$20,000+ $10,000 · ($k$ − 1)
Market (salvage) value	$10,000
Life	5 years

5-54. The American Pharmaceutical Company (APC) has a policy that all capital investments must have a four-year or less discounted payback period in order to be considered for funding. The MARR at APC is 8% per year. Is the above project able to meet this benchmark for funding? (5.8)

End of Year	Cash Flow
0	−$275,000
1	−$35,000
2	$55,000
3	$175,000
4	$250,000
5	$350,000
6–10	$100,000

5-55. The International Parcel Service has installed a new radio frequency identification system to help reduce the number of packages that are incorrectly delivered. The capital investment in the system is $65,000, and the projected annual savings are tabled below. The system's market value at the EOY five is negligible, and the MARR is 18% per year.

End of Year	Savings
1	$25,000
2	30,000
3	30,000
4	40,000
5	46,000

a. What is the FW of this investment?

b. What is the IRR of the system?

c. What is the discounted payback period for this investment?

5-56. The Fischer-Tropsch (F-T) process was developed in Germany in 1923 to convert synthesis gas (i.e., a mixture of hydrogen and carbon monoxide) into liquid with some gaseous hydrocarbons. Interestingly, the F-T process was used in World War II to make gasoline and other fuels. (5.8)

a. If the U.S. military can save one billion gallons per year of foreign oil by blending its jet fuel with F-T products, how much can the government afford to invest (through tax breaks and subsidies) in the F-T industry in the United States? The government's MARR is 7% per year, the study period is 40 years, and one gallon of jet fuel costs $2.50.

b. If Congress appropriates $25 billion to support the F-T process industry, what is the simple payback period?

5-57. An investment of $100,000 is required in an existing manufacturing concern, whose annual revenue is expected to be $40,000 for the next five years. Out of this revenue, the yearly expenditure on maintenance is $4,000. The salvage value of additional equipment installed is zero after five years. The company is willing to increase investment if the minimum rate of return is 10%. Using the IRR, determine whether the investment is desirable. (5.12)

5-58. A remotely situated fuel cell has an installed cost of $2,000 and will reduce existing surveillance expenses by $350 per year for eight years. The border security agency's MARR is 10% per year. (5.6)

a. What is the minimum salvage (market) value after eight years that makes the fuel cell worth purchasing?

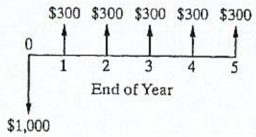
b. What is the fuel cell's IRR if the salvage value is negligible?

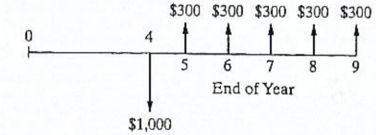
5-59. In your own words, explain to your grandmother why the values of various government bonds go up when interest rates in the U.S. economy drop. (5.3)

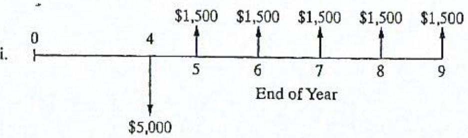
5-60.

a. Calculate the IRR for each of the three cash-flow diagrams that follow. Use EOY zero for (i) and EOY four for (ii) and (iii) as the reference points in time. What can you conclude about "reference year shift" and "proportionality" issues of the IRR method?

b. Calculate the PW at MARR = 10% per year at EOY zero for (i) and (ii) and EOY four for (ii) and (iii). How do the IRR and PW methods compare?

i.


ii.


iii.


5-61. A group of private investors borrowed $30 million to build 300 new luxury apartments near a large university. The money was borrowed at 6% annual interest, and the loan is to be repaid in equal annual amounts (principal and interest) over a 40-year period. Annual operating, maintenance, and insurance expenses are estimated to be $4,000 per apartment, and these expenses are incurred independently of the occupancy rate for the apartments. The rental fee for each apartment will be $12,000 per year, and the worst-case occupancy rate is projected to be 80%. **(5.5)**

a. How much profit (or loss) will the investors make each year with 80% occupancy?

b. Repeat Part (a) when the occupancy rate is 95%.

5-62. A hospital wants to buy a new MRI machine for $400,000. The annual revenue from the machine is estimated at $110,000 per year while maintenance costs per year are calculated to be $20,000. The salvage value at the end of the machine's five-year operational life is $100,000. You have been asked to determine the IRR of this project and to make a recommendation regarding the proposed purchase. The hospital's MARR is 20% per year. **(5.12)**

5-63. *Extended Learning Exercise* A company is producing a high-volume item that sells for $0.75 per unit. The variable production cost is $0.30 per unit. The company is able to produce and sell 10,000,000 items per year when operating at full capacity.

The critical attribute for this product is weight. The target value for weight is 1,000 grams, and the specification limits are set at ±50 grams. The filling machine used to dispense the product is capable of weights following a normal distribution with an average (μ) of 1,000 grams and a standard deviation (σ) of 40 grams. Because of the large standard deviation (with respect to the specification limits), 21.12% of all units produced are not within the specification limits. (They either weigh less than 950 grams or more than 1,050 grams.) This means that 2,112,000 out of 10,000,000 units produced are nonconforming and cannot be sold without being reworked.

Assume that nonconforming units can be reworked to specification at an additional fixed cost of $0.10 per unit. Reworked units can be sold for $0.75 per unit. It has been estimated that the demand for this product will remain at 10,000,000 units per year for the next five years.

To improve the quality of this product, the company is considering the purchase of a new filling machine. The new machine will be capable of dispensing the product with weights following a normal distribution with $\mu = 1{,}000$ grams and $\sigma = 20$ grams. As a result, the percent of nonconforming units will be reduced to 1.24% of production. The new machine will cost $710,000 and will last for at least five years. At the end of five years, this machine can be sold for $100,000.

a. If the company's MARR is 15% per year, is the purchase of the new machine to improve quality (reduce variability) economically attractive? Use the AW method to make your recommendation.

b. Compute the IRR, simple payback period, and discounted payback period of the proposed investment.

c. What other factors, in addition to reduced total rework costs, may influence the company's decision about quality improvement?

Spreadsheet Exercises

5-64. Jane Roe's plan is to accumulate $250,000 in her personal savings account by the time she retires at 60. The longer she stalls on getting started, the tougher it will be to meet her objective. Jane is now 25 years old. Create a spreadsheet to complete the following table to show how much Jane will have to save each year (assuming she waits N years to start saving) to have $250,000 in her account when she is 60. Comment on the pattern you observe in the table. **(5.4)**

	Interest Rate per Year		
	4%	8%	12%
N = 5			
N = 10			
N = 15			
N = 20			

5-65. An investor has a principal amount of $P. If he desires a payout (return) of 0.1P each year, how many years will it take to deplete an account that earns 8% per year?

$$0.1P = P(A/P, 8\%, N), \text{ so } N \cong 21 \text{ years.}$$

A payout duration table can be constructed for select payout percentages and compound interest rates. Complete the following table. (Note: table entries are years.) Summarize your conclusions about the pattern observed in the table. **(5.5)**

		Interest Rate per Year			
		4%	6%	8%	10%
Payout per Year (% of principal)	10%			20.9	
	20%				
	30%				

5-66. Refer to Example 5-13. Create a single spreadsheet that calculates PW, FW, AW, IRR, and ERR for the proposed investment. Assume that ϵ = MARR = 20% per year. Does your recommendation change if the MARR decreases to 18%? Increases to 22%? **(5.6, 5.7)**

5-67. A certain medical device will result in an estimated $15,000 reduction in hospital labor expenses during its first year of operation. Labor expenses (and thus savings) are projected to increase at a rate of 7% per year after the first year. Additional operating expenses for the device (maintenance, electric power, etc.) are $3,500 annually, and they increase by $250 per year thereafter (i.e., $3,750 in year two and so on). It is anticipated that the device will last for 10 years and will have no market value at that time. If the MARR is 10% per year, how much can the hospital afford to pay for this device? Use an Excel spreadsheet in your solution. **(5.3)**

5-68. Refer to Problem 5-61. Develop a spreadsheet to investigate the sensitivity of annual profit (loss) to changes in the occupancy rate and the annual rental fee. **(5.5)**

Case Study Exercises

5-69. Suppose that the industrial engineers are able to increase the average utilization rate of the CVD tool from 80% to 90%. What is the projected impact of this 10% increase on the PW of the project? **(5.9)**

5-70. Suppose that the mechanical engineer in the plant has retrofitted the CVD tool so that it now produces 15 wafers per hour. What is the new breakeven point? **(5.9)**

5-71. Suppose that the average utilization of the CVD tool increased to 90%; however, the operating hours of the fabrication plant were decreased from 168 hours to 150 hours. What are the corresponding impacts on the PW and breakeven point, respectively? **(5.9)**

FE Practice Problems

5-72. Doris Wade purchased a condominium for $50,000 in 1987. Her down payment was $20,000. She financed the remaining amount as a $30,000, 30-year mortgage at 7%, compounded monthly. Her monthly payments are $200. It is now 2007 (20 years later) and Doris has sold the condominium for $100,000, immediately after making her 240th payment on the unit. Her effective annual internal rate of return on this investment is closest to which answer below? **(5.6)**

(a) 3.6% (b) 8.5% (c) 5.3% (d) 1.5%

5-73. Elin purchased a used car for $10,000. She wrote a check for $2,000 as a down payment for the car and financed the $8,000 balance. The annual percentage rate (APR) is 9% compounded monthly, and the loan is to be repaid in equal monthly installments over the next four years. Which of the following is most near to Elin's monthly car payment? **(5.5)**

(a) $167 (b) $172 (c) $188
(d) $200 (e) $218

5-74. A specialized automatic machine costs $300,000 and is expected to save $111,837.50 per year while in operation. Using a 12% interest rate, what is the discounted payback period? **(5.8)**

(a) 4 (b) 5 (c) 6
(d) 7 (e) 8

5-75. Street lighting fixtures and their sodium vapor bulbs for a two-block area of a large city need to be installed at a first cost (investment cost) of $120,000. Annual maintenance expenses are expected to be $6,500 for the first 20 years and $8,500 for each year thereafter. The lighting will be needed for an indefinitely long period of time. With an interest rate of 10% per year, what is the capitalized cost of this project (choose the closest answer below)? **(5.3)**

(a) $178,313 (b) $188,000 (c) $202,045
(d) $268,000

5-76. What is the IRR for the following cash flow? **(5.6)**

Year End	0	1	2	3	4
Cash Flow ($)	−2,500	1,000	1,000	1,000	1,000

(a) 12.95% (b) 11.95% (c) 4.36%
(d) 10.05% (e) 11.05%

5-77. A bond has a face value of $1,000, is redeemable in eight years, and pays interest of $100 at the end of each of the eight years. If the bond can be purchased for $981, what is the rate of return if the bond is held until maturity? **(5.3)**

(a) 10.65% (b) 12.65% (c) 10.35%
(d) 11.65%

5-78. If you invest $5,550 in a long-term venture, you will receive $6,532 per year forever. Assuming your interest rate is 10% per year, what is the capitalized worth of your investment? Choose the closest answer below. **(5.3)**

(a) $4,327 (b) $5,977 (c) $5,819
(d) $6,103

5-79. What is the equivalent AW of a two-year contract that pays $2,000 at the beginning of the first month and increases by $200 for each month thereafter? MARR = 12% compounded monthly. **(5.5)**

(a) $10,616 (b) $131,982 (c) $5,511
(d) $4,205 (e) $134,649

5-80. A new machine was bought for $9,000 with a life of six years and no salvage value. Its annual operating costs were as follows:

$7,000, $7,350, $7,717.50, …, $8,933.97.

If the MARR = 12%, what was the annual equivalent cost of the machine? **(5.5)**

(a) $7,809 (b) $41,106 (c) $9,998
(d) $2,190 (e) $9,895

5-81. A town in Wyoming wants to drill a geothermal well to provide district heating steam and hot water for its businesses and residences. After government subsidies, the capital investment for the well is $500,000, and the geothermal well will reduce natural gas consumption for steam and hot water production by $50,000 per year. The salvage value of the well is negligible. The simple payback period for this well is 10 years. If the MARR of the town is 8% per year and the life of the geothermal well is 25 years, what is the IRR for this project? Choose the closest answer below. **(5.6)**

(a) 6.2% (b) 9.1% (c) 8.8%
(d) 10.3%

5-82. An automobile dealership offers a car with $0 down payment, $0 first month's payment, and $0 due at signing. The monthly payment, starting at the end of month two, is $386 and there are a total of 38 payments. If the APR is 12% compounded monthly, the negotiated sales prices is closest to which answer below? **(5.3)**

(a) $14,668 (b) $12,035 (c) $12,415
(d) $13,175

5-83. A 2,000 square foot house in New Jersey costs $1,725 each winter to heat with its existing oil-burning furnace. For an investment of $5,000, a natural gas furnace can be installed, and the winter heating bill is estimated to be $1,000. If the homeowner's MARR is 6% per year, what is the discounted payback period of this proposed investment? **(5.8)**

(a) 7 years (b) 8 years (c) 9 years
(d) 10 years

Appendix 5-A The Multiple Rate of Return Problem with the IRR Method

Whenever the IRR method is used and the cash flows reverse sign (from net cash outflow to net cash inflow or the opposite) more than once over the study period, we should be aware of the rather remote possibility that either no interest rate or multiple interest rates may exist. According to Descartes' rule of signs, the *maximum* number of possible IRRs in the (−1, ∞) interval for any given project is equal to the number of cash flow sign reversals during the study period. Nordstrom's criterion says if there is only one sign change in the

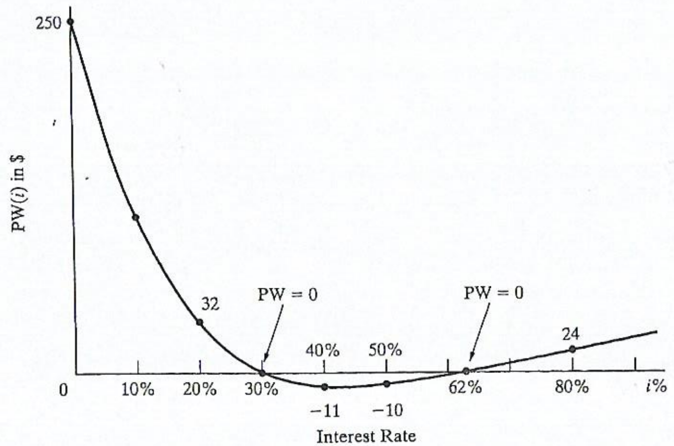
cumulative cash flows over time, a unique interest rate exists. Two or more sign changes in the cumulative cash flow indicates the possibility of multiple interest rates. Descartes' rule of signs and Nordstrom's criterion guarantee a unique internal rate of return if there is only one sign change in the cash-flow sequence and in the cumulative cash flows over time, respectively. *The simplest way to check for multiple IRRs is to plot equivalent worth (e.g., PW) against the interest rate.* If the resulting plot crosses the interest rate axis more than once, multiple IRRs are present and another equivalence method is recommended for determining project acceptability.

As an example, consider the following project for which the IRR is desired.

EXAMPLE 5-A-1

Plot the PW versus interest rate for the following cash flows. Are there multiple IRRs? If so, what do they mean? Notice there are two sign changes in net cash flows (positive, negative, positive), so at most two IRRs exist for this situation. Nordstrom's criterion also suggests a maximum of two interest rates because there are two sign changes in cumulative cash flow.

Year, k	Net Cash Flow	i%	PW(i%)	Year, k	Cumulative Cash Flow
0	$500	0	$250	0	$500
1	−1,000	10	105	1	−500
2	0	20	32	2	−500
3	250	30	~0	3	−250
4	250	40	−11	4	0
5	250	62	~0	5	250
		80	24		



Plot for Example 5-A-1

Thus, the PW of the net cash flows equals zero at interest rates of about 30% and 62%, so multiple IRRs do exist. Whenever there are multiple IRRs, which is seldom, it is likely that none are correct.

In this situation, the ERR method (see Section 5.7) could be used to decide whether the project is worthwhile. Or, we usually have the option of using an equivalent worth method. In Example 5-A-1, if the external reinvestment rate (∈) is 10% per year, we see that the ERR is 12.4%.

$$\$1{,}000(P/F, 10\%, 1)(F/P, i'\%, 5) = \$500(F/P, 10\%, 5) + \$250(F/A, 10\%, 3)$$

$$(P/F, 10\%, 1)(F/P, i', 5) = 1.632$$

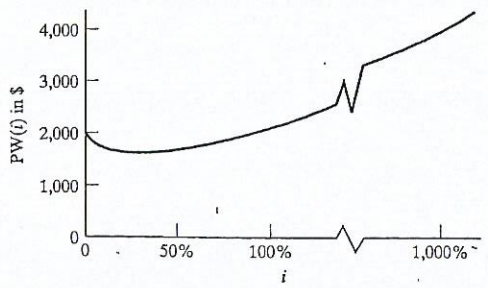
$$i' = 0.124 \; (12.4\%).$$

In addition, PW(10%) = $105, so both the ERR and PW methods indicate that this project is acceptable when the MARR is 10% per year.

EXAMPLE 5-A-2

Use the ERR method to analyze the cash-flow pattern shown in the accompanying table. The IRR is indeterminant (none exists), so the IRR is not a workable procedure. The external reinvestment rate (∈) is 12% per year, and the MARR equals 15%.

Year	Cash Flows
0	$5,000
1	−7,000
2	2,000
3	2,000



Solution

The ERR method provides this result:

$$\$7{,}000(P/F, 12\%, 1)(F/P, i'\%, 3) = \$5{,}000(F/P, 12\%, 3) + \$2{,}000(F/P, 12\%, 1) + \$2{,}000$$

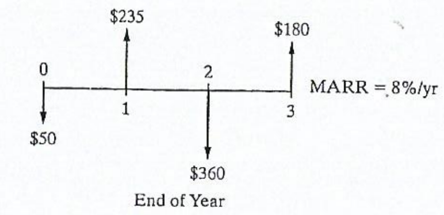
$$(F/P, i', 3) = 1.802$$

$$i' = 21.7\%.$$

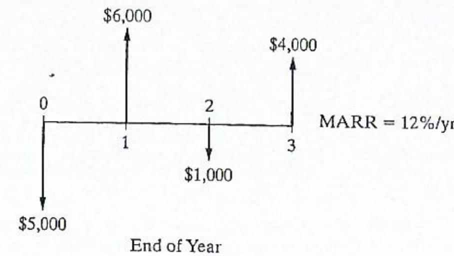
Thus, the ERR is greater than the MARR. Hence, the project having this cash-flow pattern would be acceptable. The PW at 15% is equal to $1,740.36, which confirms the acceptability of the project.

Appendix 5-A Problems

5-A-1. Use the ERR method with ∈ = 8% per year to solve for a unique rate of return for the following cash-flow diagram. How many IRRs (the maximum) are suggested by Descartes' rule of signs?



5-A-2. Apply the ERR method with ∈ = 12% per year to the following series of cash flows. Is there a single, unique IRR for these cash flows? What is the maximum number of IRRs suggested by Nordstrom's criterion?



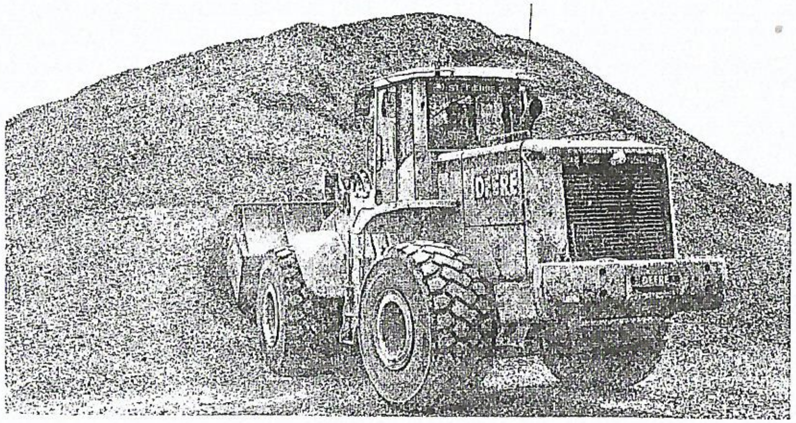
5-A-3. Are there multiple IRRs for the following cash-flow sequence? How many are possible according to Descartes' rule of signs? If ∈ = 10% per year, what is the ERR for the cash flows of this project? Let MARR = 10% per year.

EOY	0	1	2	3	4	5	6	7	8	9	10
Cash Flow ($)	120	90	60	30	−1,810	600	500	400	300	200	100

5-A-4. Buildings that are constructed to be environmentally responsible are referred to as "green buildings." They cut down on energy use, increase water efficiency, improve indoor air quality and use recycled construction materials wisely. According to recent studies, money spent on a green building will pay for itself ten times over the expected 50-year life of the building. Are there multiple internal rates of return for this situation? If so, What are they?

CHAPTER 6

Comparison and Selection among Alternatives



Our aim in Chapter 6 is to compare capital investment alternatives when the time value of money is a key influence.

Alternatives for Waste Storage

A large electric utility company is considering two methods for containing and storing its coal combustion by-products (fly ash). One method is wet slurry storage, and the second method is dry storage of the fly ash. The company will adopt one of these methods for all 28 fly ash impoundments at its seven coal-fired power plants. Wet storage has an initial capital investment of $2 billion, followed by annual maintenance expenses of $300 million over the 10-year life of the method. Dry storage has a $2.5 billion capital investment and $150 million per year annual upkeep expenditures over its 7-year life. If the utility's MARR is 10% per year, which method of fly ash storage should be selected assuming an indefinitely long study period? In Chapter 6, investment problems of this type will be considered. We will return to this problem in Example 6-9.

Rule 1: Never lose money. Rule 2: Never forget Rule 1.

—Warren Buffet (2004)

6.1 Introduction

Most engineering projects can be accomplished by more than one feasible design alternative. When the selection of one of these alternatives excludes the choice of any of the others, the alternatives are called *mutually exclusive*. Typically, the alternatives being considered require the investment of different amounts of capital, and their annual revenues and costs may vary. Sometimes the alternatives may have different useful lives. The fundamental question is "do the added benefits from a more-expensive alternative bring a positive return relative to the added costs?"

A seven-step procedure for accomplishing engineering economy studies was discussed in Chapter 1. In this chapter, we address Step 5 (analysis and comparison of the feasible alternatives) and Step 6 (selection of the preferred alternative) of this procedure, and we compare mutually exclusive alternatives on the basis of economic considerations alone.

Five of the basic methods discussed in Chapter 5 for analyzing cash flows are used in the analyses in this chapter [present worth (PW), annual worth (AW), future worth (FW), internal rate of return (IRR), and external rate of return (ERR)]. These methods provide a *basis for economic comparison* of alternatives for an engineering project. When correctly applied, these methods result in the correct selection of a preferred alternative from a set of mutually exclusive alternatives. The comparison of mutually exclusive alternatives by means of the benefit–cost ratio method is discussed in Chapter 10.

6.2 Basic Concepts for Comparing Alternatives

Principle 1 (Chapter 1) emphasized that a choice (decision) is among alternatives. Such choices must incorporate the fundamental purpose of capital investment; namely, to obtain at least the minimum attractive rate of return (MARR) for each dollar invested. In practice, there are usually a limited number of feasible alternatives to consider for an engineering project. The problem of deciding which mutually exclusive alternative should be selected is made easier if we adopt this rule based on Principle 2 (focus on the differences) in Chapter 1: *The alternative that requires the minimum investment of capital and produces satisfactory functional results will be chosen unless the incremental capital associated with an alternative having a larger investment can be justified with respect to its incremental benefits.*

Under this rule, we consider the acceptable alternative that requires the least investment of capital to be the *base alternative*. The investment of additional capital over that required by the base alternative usually results in increased capacity, increased quality, increased revenues, decreased operating expenses, or increased life. Therefore, before additional money is invested, it must be shown that each avoidable increment of capital can pay its own way relative to other available investment opportunities.

In summary, *if* the extra benefits obtained by investing additional capital are better than those that could be obtained from investment of the same capital elsewhere in the company at the MARR, the investment should be made.

If this is not the case, we obviously would not invest more than the minimum amount of capital required, and we may even do nothing at all. Stated simply, our rule will keep as much capital as possible invested at a rate of return equal to or greater than the MARR.

6.2.1 Investment and Cost Alternatives

This basic policy for the comparison of mutually exclusive alternatives can be demonstrated with two examples. The *first example* involves an investment project situation. Alternatives A and B are two mutually exclusive *investment alternatives* with estimated net cash flows,* as shown. *Investment alternatives are those with initial (or front-end) capital investment(s) that produce positive cash flows from increased revenue, savings through reduced costs, or both.* The useful life of each alternative in this example is four years.

	Alternative		
	A	B	$\Delta(B-A)$
Capital investment	−$60,000	−$73,000	−$13,000
Annual revenues less expenses	22,000	26,225	4,225

The cash-flow diagrams for Alternatives A and B, and for the year-by-year differences between them (i.e., B minus A), are shown in Figure 6-1. These diagrams typify those for investment project alternatives. In this first example, at MARR = 10% per year, the PW values are

$$\text{PW}(10\%)_A = -\$60{,}000 + \$22{,}000(P/A, 10\%, 4) = \$9{,}738,$$

$$\text{PW}(10\%)_B = -\$73{,}000 + \$26{,}225(P/A, 10\%, 4) = \$10{,}131.$$

Since the PW_A is greater than zero at i = MARR, Alternative A is the base alternative and would be selected *unless* the additional (incremental) capital associated with Alternative B ($13,000) is justified. In this case, Alternative B is preferred to A because it has a greater PW value. Hence, *the extra benefits obtained by investing the additional $13,000 of capital in B* (diagram 3, Figure 6-1) have a PW of $10,131 − $9,738 = $393. That is,

$$\text{PW}(10\%)_{\text{Diff}} = -\$13{,}000 + \$4{,}225(P/A, 10\%, 4) = \$393,$$

and the additional capital invested in B is justified.

* In this book, the terms *net cash flow* and *cash flow* will be used interchangeably when referring to periodic cash inflows and cash outflows for an alternative.

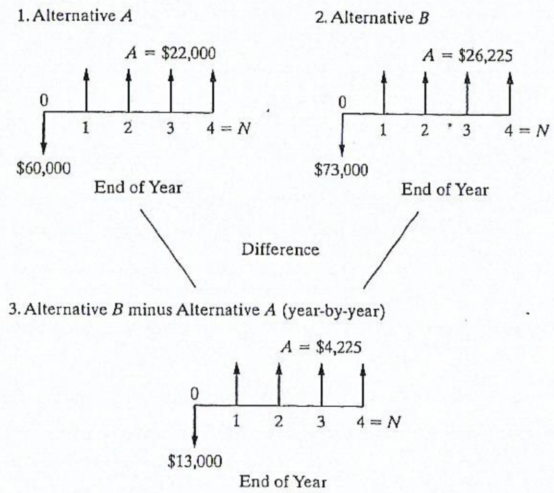


Figure 6-1 Cash-Flow Diagrams for Alternatives A and B and Their Difference

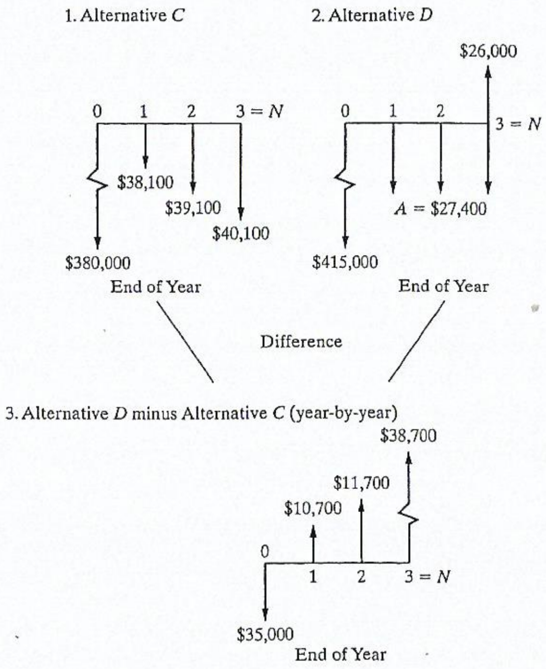
The *second example* involves a cost project situation. Alternatives C and D are two mutually exclusive *cost alternatives* with estimated net cash flows, as shown, over a three-year life. *Cost alternatives are those with all negative cash flows, except for a possible positive cash-flow element from disposal of assets at the end of the project's useful life.* This situation occurs when the organization must take some action, and the decision involves the most economical way of doing it (e.g., the addition of environmental control capability to meet new regulatory requirements). It also occurs when the expected revenues are the same for each alternative.

	Alternative		
End of Year	C	D	$\Delta(D-C)$
0	−$380,000	−$415,000	−$35,000
1	−38,100	−27,400	10,700
2	−39,100	−27,400	11,700
3	−40,100	−27,400	12,700
3[a]	0	26,000	26,000

[a] Market value.

The cash-flow diagrams for Alternatives C and D, and for the year-by-year differences between them (i.e., D minus C), are shown in Figure 6-2. These diagrams typify those for cost project alternatives. In this "must take action"

Figure 6-2 Cash-Flow Diagrams for Alternatives C and D and Their Difference



situation, Alternative C, which has the lesser capital investment, is automatically the base alternative and would be selected *unless* the additional (incremental) capital associated with Alternative *D* ($35,000) is justified. With the greater capital investment, Alternative *D* in this illustration has smaller annual expenses. Otherwise, it would not be a feasible alternative. (It would not be logical to invest more capital in an alternative without obtaining additional revenues or savings.) Note, in diagram 3, Figure 6-2, that the difference between two feasible cost alternatives is an investment alternative.

In this second example, at MARR = 10% per year, the $\text{PW}(10\%)_C = -\$477{,}077$ and the $\text{PW}(10\%)_D = -\$463{,}607$. Alternative *D* is preferred to C because it has the less negative PW (minimizes costs). Hence, *the lower annual expenses obtained by investing the additional $35,000 of capital in Alternative D* have a PW of $-\$463{,}607 - (-\$477{,}077) = \$13{,}470$. That is, the $\text{PW}(10\%)_{\text{Diff}} = \$13{,}470$, and the additional capital invested in Alternative *D* is justified.

6.2.2 Ensuring a Comparable Basis

Each feasible, mutually exclusive alternative selected for detailed analysis meets the functional requirements established for the engineering project (Section 1.3.2). Differences among the alternatives, however, may occur in many forms. Ensuring

a comparable basis for their analysis requires that any economic impacts of these differences be included in the estimated cash flows for the alternatives (as well as comparing them over the same analysis period—see Section 6.3). Otherwise, the wrong design alternative may be selected for implementing the project. The following are examples of the types of differences that may occur:

1. Operational performance factors such as output capacity, speed, thrust, heat dissipation rate, reliability, fuel efficiency, setup time, and so on
2. Quality factors such as the number of defect-free (nondefective) units produced per period or the percentage of defective units (reject rate)
3. Useful life, capital investment required, revenue changes, various annual expenses or cost savings, and so on

This list of examples could be expanded. The specific differences, however, for each engineering project and its design alternatives must be identified. Then, the cash-flow estimates for the alternatives must include the economic impact of these differences. *This is a fundamental premise for comparing alternatives in Chapter 6 and in the chapters that follow.*

Two rules were given in Section 2.4 for facilitating the correct analysis and comparison of mutually exclusive alternatives when the time value of money is not a factor (present economy studies). For convenience, these rules are repeated here and *extended to account for the time value of money:*

Rule 1: When revenues and other economic benefits are present and vary among the alternatives, choose the alternative that *maximizes* overall profitability. That is, select the alternative that has the greatest positive equivalent worth at i = MARR and satisfies all project requirements.

Rule 2: When revenues and other economic benefits are *not* present *or* are constant among the alternatives, consider only the costs and select the alternative that *minimizes* total cost. That is, select the alternative that has the least negative equivalent worth at i = MARR and satisfies all project requirements.

6.3 The Study (Analysis) Period

The study (analysis) period, sometimes called the *planning horizon*, is the selected time period over which mutually exclusive alternatives are compared. The determination of the study period for a decision situation may be influenced by several factors—for example, the service period required, the useful life* of the shorter-lived alternative, the useful life of the longer-lived alternative, and company policy. *The key point is that the selected study period must be appropriate for the decision situation under investigation.*

* The useful life of an asset is the period during which it is kept in productive use in a trade or business.

The useful lives of alternatives being compared, relative to the selected study period, can involve two situations:

1. Useful lives are the same for all alternatives and equal to the study period.
2. Useful lives are unequal among the alternatives, and at least one does not match the study period.

Unequal lives among alternatives somewhat complicate their analysis and comparison. To conduct engineering economy analyses in such cases, we adopt the rule of comparing mutually exclusive alternatives over the same period of time.

The repeatability assumption and the coterminated assumption are the two types of assumptions used for these comparisons.

The *repeatability assumption* involves two main conditions:

1. The study period over which the alternatives are being compared is either indefinitely long or equal to a common multiple of the lives of the alternatives.
2. The economic consequences that are estimated to happen in an alternative's initial useful life span will also happen in all succeeding life spans (replacements).

Actual situations in engineering practice seldom meet both conditions. This has tended to limit the use of the repeatability assumption, except in those situations where the difference between the AW of the first life cycle and the AW over more than one life cycle of the assets involved is quite small.*

The *coterminated assumption* uses a finite and identical study period for all alternatives. This planning horizon, combined with appropriate adjustments to the estimated cash flows, puts the alternatives on a common and comparable basis. For example, if the situation involves providing a service, the same time period requirement applies to each alternative in the comparison. To force a match of cash-flow durations to the cotermination time, adjustments (based on additional assumptions) are made to cash-flow estimates of project alternatives having useful lives different from the study period. For example, if an alternative has a useful life shorter than the study period, the estimated annual cost of contracting for the activities involved might be assumed and used during the remaining years. Similarly, if the useful life of an alternative is longer than the study period, a reestimated market value is normally used as a terminal cash flow at the end of a project's coterminated life.

* T. G. Eschenbach and A. E. Smith, "Violating the Identical Repetition Assumption of EAC," *Proceedings, International Industrial Engineering Conference* (May 1990), The Institute of Industrial Engineers, Norcross, GA, pp. 99–104.

6.4 Useful Lives Are Equal to the Study Period

When the useful life of an alternative is equal to the selected study period, adjustments to the cash flows are not required. In this section, we discuss the comparison of mutually exclusive alternatives, using equivalent-worth methods and rate-of-return methods when the useful lives of all alternatives are equal to the study period.

6.4.1 Equivalent-Worth Methods

In Chapter 5, we learned that the equivalent-worth methods convert all relevant cash flows into equivalent present, annual, or future amounts. When these methods are used, consistency of alternative selection results from this equivalency relationship. Also, the economic ranking of mutually exclusive alternatives will be the same when using the three methods. Consider the general case of two alternatives A and B. If $\text{PW}(i\%)_A < \text{PW}(i\%)_B$, then the FW and AW analyses will result in the same preference for Alternative B.

The most straightforward technique for comparing mutually exclusive alternatives when all useful lives are equal to the study period is to determine the equivalent worth of each alternative based on total investment at $i = \text{MARR}$. Then, for investment alternatives, the one with the greatest positive equivalent worth is selected. And, in the case of cost alternatives, the one with the least negative equivalent worth is selected.

EXAMPLE 6-1 **Analyzing Investment Alternatives by Using Equivalent Worth**

Best Flight, Inc., is considering three mutually exclusive alternatives for implementing an automated passenger check-in counter at its hub airport. Each alternative meets the same service requirements, but differences in capital investment amounts and benefits (cost savings) exist among them. The study period is 10 years, and the useful lives of all three alternatives are also 10 years. Market values of all alternatives are assumed to be zero at the end of their useful lives. If the airline's MARR is 10% per year, which alternative should be selected in view of the cash-flow diagrams shown on page 256?

Solution by the PW Method

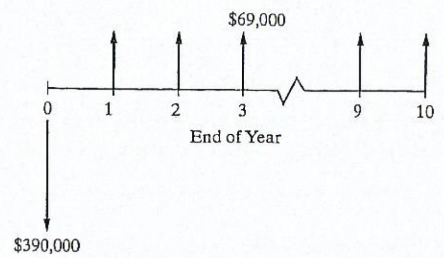
$$\text{PW}(10\%)_A = -\$390{,}000 + \$69{,}000(P/A, 10\%, 10) = \$33{,}977,$$

$$\text{PW}(10\%)_B = -\$920{,}000 + \$167{,}000(P/A, 10\%, 10) = \$106{,}148,$$

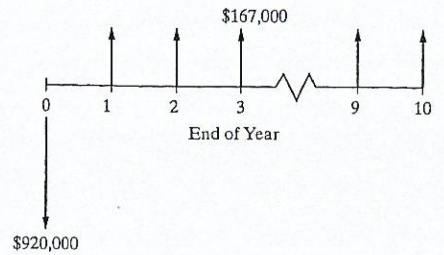
$$\text{PW}(10\%)_C = -\$660{,}000 + \$133{,}500(P/A, 10\%, 10) = \$160{,}304.$$

Based on the PW method, Alternative C would be selected because it has the largest PW value (\$160,304). The order of preference is $C \succ B \succ A$, where $C \succ B$ means C is preferred to B.

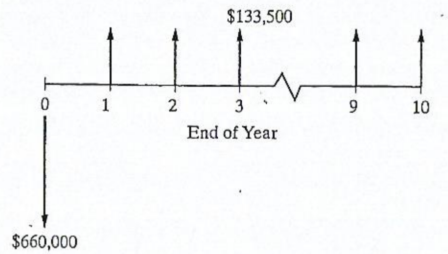
Alternative A:



Alternative B:



Alternative C:



Solution by the AW Method

$$AW(10\%)_A = -\$390{,}000(A/P, 10\%, 10) + \$69{,}000 = \$5{,}547,$$

$$AW(10\%)_B = -\$920{,}000(A/P, 10\%, 10) + \$167{,}000 = \$17{,}316,$$

$$AW(10\%)_C = -\$660{,}000(A/P, 10\%, 10) + \$133{,}500 = \$26{,}118.$$

Alternative C is again chosen because it has the largest AW value ($26,118).

Solution by the FW Method

$$FW(10\%)_A = -\$390{,}000(F/P, 10\%, 10) + \$69{,}000(F/A, 10\%, 10) = \$88{,}138,$$

$$FW(10\%)_B = -\$920{,}000(F/P, 10\%, 10) + \$167{,}000(F/A, 10\%, 10) = \$275{,}342,$$

$$FW(10\%)_C = -\$660{,}000(F/P, 10\%, 10) + \$133{,}500(F/A, 10\%, 10) = \$415{,}801.$$

Based on the FW method, the choice is again Alternative C because it has the largest FW value ($415,801). For all three methods (PW, AW, and FW) in this example, notice that C $\succ$ $B \succ A$ because of the equivalency relationship among the methods. Also, notice that Rule 1 (Section 6.2.2) applies in this example, since the economic benefits (cost savings) vary among the alternatives.

Example 6-2 and Example 6-3 illustrate the impact that estimated differences in the capability of alternatives to produce defect-free products have on the economic analysis. In Example 6-2, each of the plastic-molding presses produces the same total amount of output units, all of which are defect free. Then, in Example 6-3, each press still produces the same total amount of output units, but the percentage of defective units (reject rate) varies among the presses.

EXAMPLE 6-2

Analyzing Cost-Only Alternatives, Using Equivalent Worth

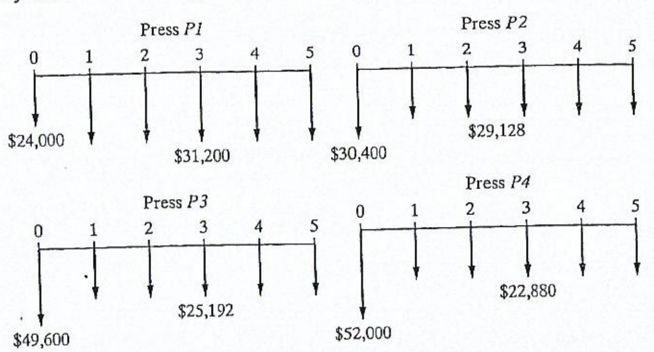
A company is planning to install a new automated plastic-molding press. Four different presses are available. The initial capital investments and annual expenses for these four mutually exclusive alternatives are as follows:

	Press			
	P1	*P2*	*P3*	*P4*
Capital investment	$24,000	$30,400	$49,600	$52,000
Useful life (years)	5	5	5	5
Annual expenses				
Power	2,720	2,720	4,800	5,040
Labor	26,400	24,000	16,800	14,800
Maintenance	1,600	1,800	2,600	2,000
Property taxes and insurance	480	608	992	1,040
Total annual expenses	$31,200	$29,128	$25,192	$22,880

Assume that each press has the same output capacity (120,000 units per year) and has no market value at the end of its useful life; the selected analysis period is five years; and any additional capital invested is expected to earn at least 10% per year. Which press should be chosen if 120,000 nondefective units per year are produced by each press and all units can be sold? The selling price is $0.375 per unit. Solve by hand and by spreadsheet.

Solution by Hand

Since the same number of nondefective units per year will be produced and sold using each press, revenue can be disregarded (Principle 2, Chapter 1). The end-of-year cash-flow diagrams of the four presses are:



The preferred alternative will minimize the equivalent worth of total costs over the five-year analysis period (Rule 2, page 253). That is, the four alternatives can be compared as cost alternatives. The PW, AW, and FW calculations for Alternative $P1$ are

$$\text{PW}(10\%)_{P1} = -\$24{,}000 - \$31{,}200(P/A, 10\%, 5) = -\$142{,}273,$$

$$\text{AW}(10\%)_{P1} = -\$24{,}000(A/P, 10\%, 5) - \$31{,}200 = -\$37{,}531,$$

$$\text{FW}(10\%)_{P1} = -\$24{,}000(F/P, 10\%, 5) - \$31{,}200(F/A, 10\%, 5)$$

$$= -\$229{,}131.$$

The PW, AW, and FW values for Alternatives $P2$, $P3$, and $P4$ are determined with similar calculations and shown for all four presses in Table 6-1. Alternative $P4$ minimizes all three equivalent-worth values of total costs and is the preferred alternative. The preference ranking ($P4 \succ P2 \succ P1 \succ P3$) resulting from the analysis is the same for all three methods.

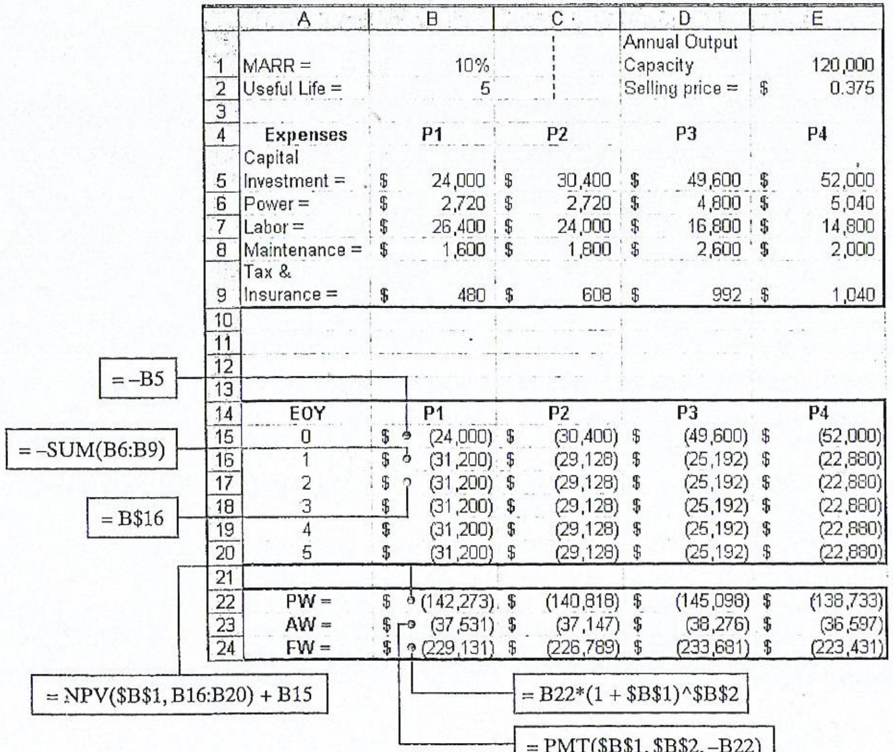
Comparison of Four Molding Presses, Using the PW, AW, and FW Methods to Minimize Total Costs

	Press (Equivalent-Worth Values)			
Method	P1	P2	P3	P4
Present worth	−$142,273	−$140,818	−$145,098	−$138,734
Annual worth	−37,531	−37,148	−38,276	−36,598
Future worth	−229,131	−226,788	−233,689	−223,431

Spreadsheet Solution

Figure 6-3 presents a spreadsheet solution for identifying the press that minimizes total equivalent costs. The top section of the spreadsheet displays the problem data. These data are then tabulated as total end-of-year (EOY) cash flows in the middle section of the spreadsheet. Finally, the PW, AW, and FW amounts are computed and the results displayed at the bottom of the spreadsheet. Note that the AW and FW cell formulas make use of the PW result in row 22. The results are the same (except for rounding) as those computed by hand and displayed in Table 6-1.

NOTE: This spreadsheet model allows us to easily do what the solution by hand does not—evaluate how our recommendation will change if data values change. For example, if MARR changes from 10% to 15%, then Press 2 becomes the lowest cost alternative.



	A	B	C	D	E
1	MARR =	10%		Annual Output Capacity	120,000
2	Useful Life =	5		Selling price =	$ 0.375
3					
4	Expenses	P1	P2	P3	P4
5	Capital Investment =	$ 24,000	$ 30,400	$ 49,600	$ 52,000
6	Power =	$ 2,720	$ 2,720	$ 4,800	$ 5,040
7	Labor =	$ 26,400	$ 24,000	$ 16,800	$ 14,800
8	Maintenance =	$ 1,600	$ 1,800	$ 2,600	$ 2,000
9	Tax & Insurance =	$ 480	$ 608	$ 992	$ 1,040
10					
11					
12					
13					
14	EOY	P1	P2	P3	P4
15	0	$ (24,000)	$ (30,400)	$ (49,600)	$ (52,000)
16	1	$ (31,200)	$ (29,128)	$ (25,192)	$ (22,880)
17	2	$ (31,200)	$ (29,128)	$ (25,192)	$ (22,880)
18	3	$ (31,200)	$ (29,128)	$ (25,192)	$ (22,880)
19	4	$ (31,200)	$ (29,128)	$ (25,192)	$ (22,880)
20	5	$ (31,200)	$ (29,128)	$ (25,192)	$ (22,880)
21					
22	PW =	$ (142,273)	$ (140,818)	$ (145,098)	$ (138,733)
23	AW =	$ (37,531)	$ (37,147)	$ (38,276)	$ (36,597)
24	FW =	$ (229,131)	$ (226,789)	$ (233,681)	$ (223,431)

Figure 6-3 Spreadsheet Solution, Example 6-2

EXAMPLE 6-3

Analyzing Alternatives with Different Reject Rates

Consider the four plastic molding presses of Example 6-2. Suppose that each press is still capable of producing 120,000 total units per year, but the estimated reject rate is different for each alternative. This means that the expected revenue will differ among the alternatives since only nondefective units can be sold. The data for the four presses are summarized below. The life of each press (and the study period) is five years.

	Press			
	P1	*P2*	*P3*	*P4*
Capital investment	\$24,000	\$30,400	\$49,600	\$52,000
Total annual expenses	\$31,200	\$29,128	\$25,192	\$22,880
Reject rate	8.4%	0.3%	2.6%	5.6%

If all nondefective units can be sold for \$0.375 per unit, which press should be chosen? Solve by hand and by spreadsheet.

Solution by Hand

In this example, each of the four alternative presses produces 120,000 units per year, but they have different estimated reject rates. Therefore, the number of nondefective output units produced and sold per year, as well as the annual revenues received by the company, varies among the alternatives. But the annual expenses are assumed to be unaffected by the reject rates. In this situation, the preferred alternative will maximize overall profitability (Rule 1, Section 6.2.2). That is, the four presses need to be compared as investment alternatives. The PW, AW, and FW calculations for Alternative *P4* are given below:

$$\text{PW}(10\%)_{P4} = -\$52{,}000 + [(1-0.056)(120{,}000)(\$0.375) - \$22{,}880](P/A, 10\%, 5)$$
$$= \$22{,}300,$$

$$\text{AW}(10\%)_{P4} = -\$52{,}000(A/P, 10\%, 5) + [(1-0.056)(120{,}000)(\$0.375) - \$22{,}880]$$
$$= \$5{,}882,$$

$$\text{FW}(10\%)_{P4} = -\$52{,}000(F/P, 10\%, 5)$$
$$+ [(1-0.056)(120{,}000)(\$0.375) - \$22{,}800](F/A, 10\%, 5)$$
$$= \$35{,}914.$$

The PW, AW, and FW values for Alternatives *P1*, *P2*, and *P3* are determined with similar calculations and shown for all four alternatives in Table 6-2. Alternative *P2* maximizes all three equivalent-worth measures of overall profitability and is preferred [versus *P4* in Example 6-2]. The preference ranking ($P2 \succ P4 \succ P3 \succ P1$) is the same for the three methods but is different from the ranking in Example 6-2. The different preferred alternative and preference ranking are the result of the varying capability among the presses to produce nondefective output units.

Comparison of Four Molding Presses, Using the PW, AW, and FW Methods to Maximize Overall Profitability

	Press (Equivalent-Worth Values)			
Method	*P1*	*P2*	*P3*	*P4*
Present worth	\$13,984	\$29,256	\$21,053	\$22,300
Annual worth	3,689	7,718	5,554	5,882
Future worth	22,521	47,117	33,906	35,914

Spreadsheet Solution

Figure 6-4 displays the spreadsheet solution for identifying the preferred press when the impact of different annual revenues among the alternatives is included. The data section of the spreadsheet includes the reject rate for each press, which is used to compute expected annual revenues in row 12. The revenue values are then combined with annual expenses to arrive at the EOY net cash flows for each alternative. The resulting equivalent-worth amounts are the same (except for rounding) as those computed by hand and shown in Table 6-2.

	A	B	C	D	E
1	MARR =	10%		Annual Output Capacity	120,000
2	Useful Life =	5		Selling price =	\$ 0.375
3					
4	**Expenses**	P1	P2	P3	P4
5	Capital Investment	\$ 24,000	\$ 30,400	\$ 49,600	\$ 52,000
6	Power	\$ 2,720	\$ 2,720	\$ 4,800	\$ 5,040
7	Labor	\$ 26,400	\$ 24,000	\$ 16,800	\$ 14,800
8	Maintenance	\$ 1,600	\$ 1,800	\$ 2,600	\$ 2,000
9	Tax & Insurance	\$ 480	\$ 608	\$ 992	\$ 1,040
10					
11	Reject Rate	8.4%	0.3%	2.6%	5.6%
12	Revenue	\$ 41,220	\$ 44,865	\$ 43,830	\$ 42,480
13					
14	EOY	P1	P2	P3	P4
15	0	\$ (24,000)	\$ (30,400)	\$ (49,600)	\$ (52,000)
16	1	\$ 10,020	\$ 15,737	\$ 18,638	\$ 19,600
17	2	\$ 10,020	\$ 15,737	\$ 18,638	\$ 19,600
18	3	\$ 10,020	\$ 15,737	\$ 18,638	\$ 19,600
19	4	\$ 10,020	\$ 15,737	\$ 18,638	\$ 19,600
20	5	\$ 10,020	\$ 15,737	\$ 18,638	\$ 19,600
21					
22	PW =	\$ 13,984	\$ 29,256	\$ 21,053	\$ 22,299
23	AW =	\$ 3,689	\$ 7,718	\$ 5,554	\$ 5,883
24	FW =	\$ 22,521	\$ 47,116	\$ 33,906	\$ 35,913

= (1 – B11) *\$E\$1 *\$E\$2

= -SUM(B6:B9) + B\$12

Figure 6-4 Spreadsheet Solution, Example 6-3

6.4.2 Rate-of-Return Methods

Annual return on investment is a popular metric of profitability in the United States. When using rate-of-return methods to evaluate mutually exclusive alternatives, the best alternative produces satisfactory functional results and requires the minimum investment of capital. This is true unless a larger investment can be justified in terms of its incremental benefits and costs. Accordingly, these three guidelines are applicable to rate-of-return methods:

1. Each increment of capital must justify itself by producing a sufficient rate of return (greater than or equal to MARR) on that increment.
2. Compare a higher investment alternative against a lower investment alternative only when the latter is acceptable. The difference between the two alternatives is usually an *investment alternative* and permits the better one to be determined.
3. Select the alternative that requires the largest investment of capital, as long as the incremental investment is justified by benefits that earn at least the MARR. This maximizes equivalent worth on total investment at i = MARR.

Do not compare the IRRs of mutually exclusive alternatives (or IRRs of the differences between mutually exclusive alternatives) against those of other alternatives. Compare an IRR only against MARR (IRR $\geq$ MARR) in determining the acceptability of an alternative.

These guidelines can be implemented using the *incremental investment analysis technique* with rate-of-return methods.* First, however, we will discuss the *inconsistent ranking problem* that can occur with incorrect use of rate-of-return methods in the comparison of alternatives.

6.4.2.1 The Inconsistent Ranking Problem In Section 6.2, we discussed a small investment project involving two alternatives, A and B. The cash flow for each alternative is restated here, as well as the cash flow (incremental) difference.

	Alternative		Difference
	A	B	$\Delta(B-A)$
Capital investment	\$60,000	\$73,000	\$13,000
Annual revenues less expenses	22,000	26,225	4,225

The useful life of each alternative (and the study period) is four years. Also, assume that MARR = 10% per year. First, check to see if the sum of positive cash flows exceeds the sum of negative cash flows. This is the case here, so the IRR and PW(10%) of each alternative are calculated and shown as follows:

* The IRR method is the most celebrated time value-of-money-based profitability metric in the United States. The incremental analysis technique must be learned so that the IRR method can be correctly applied in the comparison of mutually exclusive alternatives.

Alternative	IRR	PW(10%)
A	17.3%	\$9,738
B	16.3	10,131

If, at this point, a choice were made based on maximizing the IRR of the total cash flow, Alternative A would be selected. But, based on maximizing the PW of the total investment at i = MARR, Alternative B is preferred. Obviously, here we have an inconsistent ranking of the two mutually exclusive investment alternatives.

Now that we know Alternative A is acceptable (IRR > MARR; PW at MARR > 0), we will analyze the incremental cash flow between the two alternatives, which we shall refer to as $\Delta(B-A)$. The IRR of this increment, IRR_Δ, is 11.4%. This is greater than the MARR of 10%, and the incremental investment of \$13,000 is justified. This outcome is confirmed by the PW of the increment, PW_Δ (10%), which is equal to \$393. Thus, when the IRR of the incremental cash flow is used, the rankings of A and B are consistent with that based on the PW on total investment.

The fundamental role that the incremental net cash flow, $\Delta(B-A)$, plays in the comparison of two alternatives (where B has the greater capital investment) is based on the following relationship:

Cash flow of B = Cash flow of A + Cash flow of the difference.

Clearly, the cash flow of B is made up of two parts. The first part is equal to the cash flow of Alternative A, and the second part is the incremental cash flow between A and B, $\Delta(B-A)$. Obviously, if the equivalent worth of the difference is greater than or equal to zero at i = MARR, then Alternative B is preferred. Otherwise, given that Alternative A is justified (an acceptable *base alternative*), Alternative A is preferred. It is always true that if $\text{PW}_\Delta \geq 0$, then $\text{IRR}_\Delta \geq$ MARR.

Figure 6-5 illustrates how ranking errors can occur when a selection among mutually exclusive alternatives is based wrongly on maximization of IRR on the total cash flow. When MARR lies to the left of IRR_Δ (11.4% in this case), an incorrect choice will be made by selecting an alternative that maximizes IRR. This is because the IRR method assumes reinvestment of cash flows at the calculated rate of return (17.3% and 16.3%, respectively, for Alternatives A and B in this case), whereas the PW method assumes reinvestment at MARR (10%).

Figure 6-5 shows our previous results with $\text{PW}_B > \text{PW}_A$ at MARR = 10%, even though $\text{IRR}_A > \text{IRR}_B$. Also, the figure shows how to avoid this ranking inconsistency by examining the IRR of the increment, IRR_Δ, which correctly leads to the selection of Alternative B, the same as with the PW method.

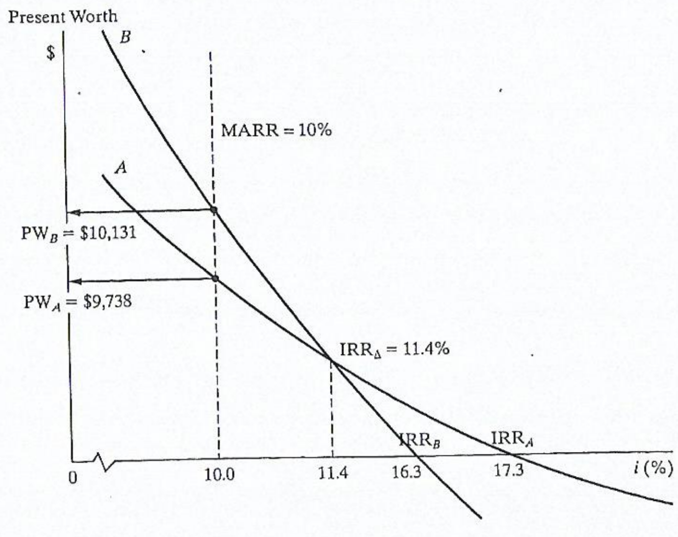


Figure 6-5 Illustration of the Ranking Error in Studies Using the IRR Method

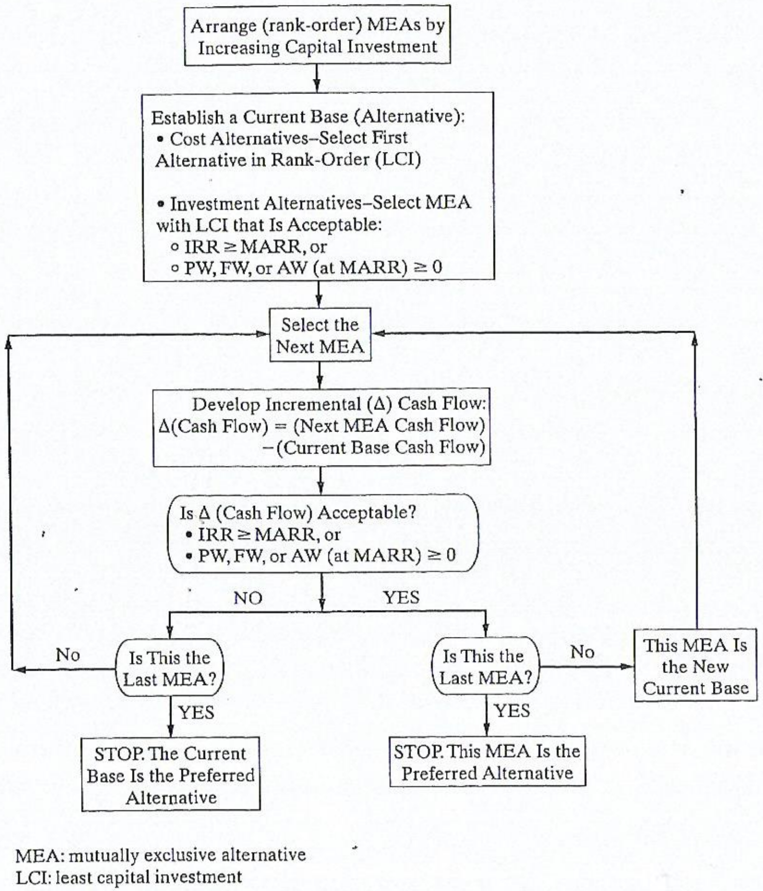
6.4.2.2 The Incremental Investment Analysis Procedure

We recommend the incremental investment analysis procedure to avoid incorrect ranking of mutually exclusive alternatives when using rate-of-return methods. We will use this procedure in the remainder of the book.

The incremental analysis procedure for the comparison of mutually exclusive alternatives is summarized in three basic steps (illustrated in Figure 6-6):

1. Arrange (rank-order) the feasible alternatives based on increasing capital investment.*
2. Establish a base alternative.
   (a) Cost alternatives—the first alternative (least capital investment) is the base.
   (b) Investment alternatives—if the first alternative is acceptable (IRR ≥ MARR; PW, FW, or AW at MARR ≥ 0), select it as the base. If the first alternative is not acceptable, choose the next alternative in order of increasing

* This ranking rule assumes a logical set of mutually exclusive alternatives. That is to say, for investment or cost alternatives, increased initial investment results in additional economic benefits, whether from added revenues, reduced costs, or a combination of both. Also, this rule assumes that for any nonconventional investment cash flow, the PW, AW, FW, or ERR analysis method would be used instead of IRR. Simply stated, a nonconventional investment cash flow involves multiple sign changes or positive cash flow at time zero, or both. For a more detailed discussion of ranking rules, see C. S. Park and G. P. Sharp-Bette, *Advanced Engineering Economy* (New York: John Wiley & Sons, 1990).



MEA: mutually exclusive alternative
LCI: least capital investment

Figure 6-6 Incremental Investment Analysis Procedure

capital investment and check the profitability criterion (PW, etc.) values. Continue until an acceptable alternative is obtained. If none is obtained, the do-nothing alternative is selected.

3. Use iteration to evaluate differences (incremental cash flows) between alternatives until all alternatives have been considered.
   (a) If the incremental cash flow between the next (higher capital investment) alternative and the current selected alternative is acceptable, choose the next alternative as the current best alternative. Otherwise, retain the last acceptable alternative as the current best.

(b) Repeat and select as the preferred alternative the last one for which the incremental cash flow was acceptable.

EXAMPLE 6-4

Incremental Analysis: Investment Alternatives

Suppose that we are analyzing the following six mutually exclusive alternatives for a small investment project, using the IRR method. The useful life of each alternative is 10 years, and the MARR is 10% per year. Also, net annual revenues less expenses vary among all alternatives, and Rule 1, Section 6.2.2, applies. If the study period is 10 years, and the market (salvage) values are zero, which alternative should be chosen? Notice that the alternatives have been *rank-ordered* from *low capital investment* to high capital investment.

	Alternative					
	A	*B*	*C*	*D*	*E*	*F*
Capital investment	$900	$1,500	$2,500	$4,000	$5,000	$7,000
Annual revenues less expenses	150	276	400	925	1,125	1,425

Solution

For each of the feasible alternatives, the IRR on the total cash flow can be computed by determining the interest rate at which the PW, FW, or AW equals zero (use of AW is illustrated for Alternative *A*):*

$$0 = -\$900(A/P, i'_A\%, 10) + \$150; \quad i'\% = ?$$

By trial and error, we determine that $i'_A\% = 10.6\%$. In the same manner, the IRRs of all the alternatives are computed and summarized:

	A	*B*	*C*	*D*	*E*	*F*
IRR on total cash flow	10.6%	13.0%	9.6%	19.1%	18.3%	15.6%

At this point, *only Alternative C is unacceptable* and can be eliminated from the comparison because its IRR is less than MARR of 10% per year. Also, *A* is the base alternative from which to begin the incremental investment analysis procedure, because it is the mutually exclusive alternative with the lowest capital investment whose IRR (10.6%) is equal to or greater than MARR (10%).

* The three steps of the incremental analysis procedure previously discussed (and illustrated in Figure 6-6) do not require the calculation of the IRR value for each alternative. In this example, the IRR of each alternative is used for illustrating common errors made with the IRR method.

Comparison of Five Acceptable Investment Alternatives Using the IRR Method (Example 6-4)

Increment Considered	*A*	$\Delta(B-A)$	$\Delta(D-B)$	$\Delta(E-D)$	$\Delta(F-E)$
Δ Capital investment	$900	$600	$2,500	$1,000	$2,000
Δ Annual revenues less expenses	$150	$126	$649	$200	$300
IRR_{Δ}	10.6%	16.4%	22.6%	15.1%	8.1%
Is increment justified?	Yes	Yes	Yes	Yes	No

This pre-analysis of the feasibility of each alternative is not required by the incremental analysis procedure. It is useful, however, when analyzing a larger set of mutually exclusive alternatives. You can immediately eliminate nonfeasible (nonprofitable) alternatives, as well as easily identify the base alternative.

As discussed in Section 6.4.2.1, it is not necessarily correct to select the alternative that maximizes the IRR on total cash flow. That is to say, Alternative *D* may not be the best choice, since *maximization of IRR does not guarantee maximization of equivalent worth on total investment at the MARR*. Therefore, to make the correct choice, we must examine each increment of capital investment to see if it will pay its own way. Table 6-3 provides the analysis of the five remaining alternatives, and the IRRs on incremental cash flows are again computed by setting $AW_{\Delta}(i') = 0$ for cash-flow differences between alternatives.

From Table 6-3, it is apparent that Alternative *E* will be chosen (not *D*) because it requires the largest investment for which the last increment of capital investment is justified. That is, we desire to invest additional increments of the $7,000 presumably available for this project as long as each avoidable increment of investment can earn 10% per year or better.

It was assumed in Example 6-4 (and in all other examples involving mutually exclusive alternatives, unless noted to the contrary) that available capital for a project *not* committed to one of the feasible alternatives is invested in some other project where it will earn an annual return equal to the MARR. Therefore, in this case, the $2,000 left over by selecting Alternative *E* instead of *F* is assumed to earn 10% per year elsewhere, which is more than we could obtain by investing it in *F*.

In sum, three errors commonly made in this type of analysis are to choose the mutually exclusive alternative (1) with the highest overall IRR on total cash flow, (2) with the highest IRR on an incremental capital investment, or (3) with the largest capital investment that has an IRR greater than or equal to the MARR. None of these criteria are generally correct. For instance, in Example 6-4, we might erroneously choose Alternative *D* rather than *E* because the IRR for the increment from *B* to *D* is 22.6% and that from *D* to *E* is only 15.1% (error 2). A more obvious error, as previously discussed, is the temptation to maximize the IRR on total cash flow and select Alternative *D* (error 1). The third error would be committed by selecting Alternative *F* for the reason that it has the largest total investment with an IRR greater than the MARR (15.6% > 10%).

EXAMPLE 6-5 Incremental Analysis: Cost-Only Alternatives

The estimated capital investment and the annual expenses (based on 1,500 hours of operation per year) for four alternative designs of a diesel-powered air compressor are shown, as well as the estimated market value for each design at the end of the common five-year useful life. The perspective (Principle 3, Chapter 1) of these cost estimates is that of the typical user (construction company, plant facilities department, government highway department, and so on). The study period is five years, and the MARR is 20% per year. One of the designs must be selected for the compressor, and each design provides the same level of service. On the basis of this information,

(a) determine the preferred design alternative, using the IRR method

(b) show that the PW method (i = MARR), using the incremental analysis procedure, results in the same decision.

Solve by hand and by spreadsheet.

	Design Alternative			
	D1	*D2*	*D3*	*D4*
Capital investment	\$100,000	\$140,600	\$148,200	\$122,000
Annual expenses	29,000	16,900	14,800	22,100
Useful life (years)	5	5	5	5
Market value	10,000	14,000	25,600	14,000

Observe that this example is a *cost-type situation with four mutually exclusive cost alternatives*. The following solution demonstrates the use of the incremental analysis procedure to compare cost alternatives and applies Rule 2 in Section 6.2.2.

Solution by Hand

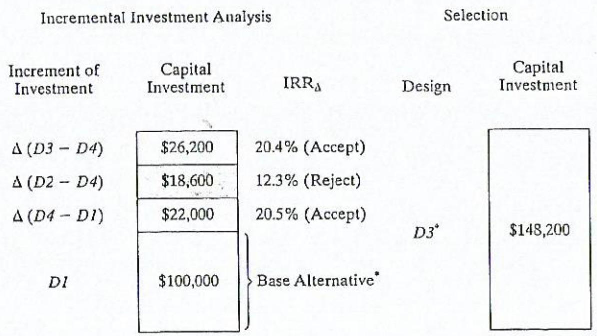
The first step is to arrange (rank-order) the four mutually exclusive cost alternatives on the basis of their increasing capital investment costs. Therefore, the order of the alternatives for incremental analysis is *D1*, *D4*, *D2*, and *D3*.

Since these are cost alternatives, the one with the least capital investment, *D1*, is the base alternative. Therefore, the base alternative will be preferred unless additional increments of capital investment can produce cost savings (benefits) that lead to a return equal to or greater than the MARR.

The first incremental cash flow to be analyzed is that between designs *D1* and *D4*, Δ(*D4* – *D1*). The results of this analysis, and of subsequent differences between the cost alternatives, are summarized in Table 6-4, and the incremental investment analysis for the IRR method is illustrated in Figure 6-7. These results show the following:

Comparison of Four Cost (Design) Alternatives Using the IRR and PW Methods with Incremental Analysis (Example 6-5)

Increment Considered	Δ(*D4* – *D1*)	Δ(*D2* – *D4*)	Δ(*D3* – *D4*)
Δ Capital investment	\$22,000	\$18,600	\$26,200
Δ Annual expense (savings)	6,900	5,200	7,300
Δ Market value	4,000	0	11,600
Useful life (years)	5	5	5
IRR_Δ	20.5%	12.3%	20.4%
Is increment justified?	Yes	No	Yes
$PW_\Delta(20\%)$	\$243	–\$3,049	\$293
Is increment justified?	Yes	No	Yes



* Since these are cost alternatives, the IRR of *D3* cannot be determined.

Figure 6-7 Representation of Capital Investment Increments and IRR on Increments Considered in Selecting Design 3 (*D3*) in Example 6-5

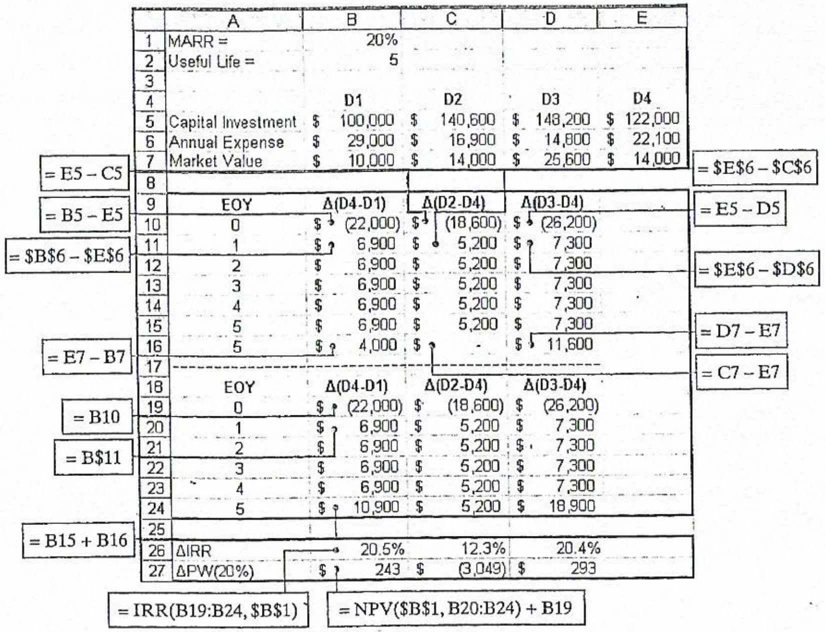
1. The incremental cash flows between the cost alternatives are, in fact, investment alternatives.
2. The first increment, Δ(*D4* – *D1*), is justified (IRR_Δ = 20.5% is greater than MARR = 20%, and $PW_\Delta(20\%)$ = \$243 > 0); the increment Δ(*D2* – *D4*) is not justified; and the last increment, Δ(*D3* – *D4*)—not Δ(*D3* – *D2*), because Design *D2* has already been shown to be unacceptable—is justified, resulting in the selection of Design *D3* for the air compressor. It is the highest investment for which each increment of investment capital is justified from the user's perspective.
3. The same capital investment decision results from the IRR method and the PW method, using the incremental analysis procedure, because *when the equivalent worth of an investment at i = MARR is greater than zero, its IRR is greater than MARR* (from the definition of the IRR; Chapter 5).

Spreadsheet Solution

Figure 6-8 shows the complete spreadsheet solution for this example. The first set of incremental EOY cash flows has two EOY five entries: one for the difference in annual expense savings and one for the difference in market value. These two values are combined in the second set of incremental EOY cash flows for direct computation of the incremental IRR and PW amounts. Note that the IRR function can handle EOY zero through EOY five cash flows as given, while the PW computation needs to add the EOY zero cash flow outside of the NPV function.

As was previously discovered via manual computation, the first increment $\Delta(D4 - D1)$ is justified; the increment $\Delta(D2 - D4)$ is not justified; and the last increment $\Delta(D3 - D4)$ is justified. Thus, *D3* is selected as the preferred compressor.

As previously mentioned, spreadsheets make it easy to answer *what if* types of questions. For example, how much would the annual expense have to be for compressor *D4* for it to become preferable to *D3*? (*Hint*: The incremental PW has to be positive.) It is easy to change the value in cell E6 by hand to bracket a small positive PW value. This corresponds to an annual expense of $22,002 for *D4*. It can also be solved quickly by using the Solver tool.



	A	B	C	D	E
1	MARR =	20%			
2	Useful Life =	5			
3					
4		D1	D2	D3	D4
5	Capital Investment	$ 100,000	$ 140,600	$ 148,200	$ 122,000
6	Annual Expense	$ 29,000	$ 16,900	$ 14,800	$ 22,100
7	Market Value	$ 10,000	$ 14,000	$ 25,600	$ 14,000
8					
9	EOY	Δ(D4-D1)	Δ(D2-D4)	Δ(D3-D4)	
10	0	$ (22,000)	$ (18,600)	$ (26,200)	
11	1	$ 6,900	$ 5,200	$ 7,300	
12	2	$ 6,900	$ 5,200	$ 7,300	
13	3	$ 6,900	$ 5,200	$ 7,300	
14	4	$ 6,900	$ 5,200	$ 7,300	
15	5	$ 6,900	$ 5,200	$ 7,300	
16	5	$ 4,000	$ -	$ 11,600	
17					
18	EOY	Δ(D4-D1)	Δ(D2-D4)	Δ(D3-D4)	
19	0	$ (22,000)	$ (18,600)	$ (26,200)	
20	1	$ 6,900	$ 5,200	$ 7,300	
21	2	$ 6,900	$ 5,200	$ 7,300	
22	3	$ 6,900	$ 5,200	$ 7,300	
23	4	$ 6,900	$ 5,200	$ 7,300	
24	5	$ 10,900	$ 5,200	$ 18,900	
25					
26	ΔIRR	20.5%	12.3%	20.4%	
27	ΔPW(20%)	$ 243	$ (3,049)	$ 293	

Figure 6-8 Spreadsheet Solution, Example 6-5

Now we turn our attention to the ERR method which was explained in Chapter 5. Also, in Appendix 5-A, the ERR method was illustrated as a substitute for the IRR method when analyzing a nonconventional investment type of cash flow. In Example 6-6, the ERR method is applied using the incremental investment analysis procedure to compare the mutually exclusive alternatives for an engineering improvement project.

EXAMPLE 6-6

Incremental Analysis Using ERR

In an automotive parts plant, an engineering team is analyzing an improvement project to increase the productivity of a flexible manufacturing center. The estimated net cash flows for the three feasible alternatives being compared are shown in Table 6-5. The analysis period is six years, and MARR for capital investments at the plant is 20% per year. Using the ERR method, which alternative should be selected? (ϵ = MARR.)

Solution

The procedure for using the ERR method to compare mutually exclusive alternatives is the same as for the IRR method. The only difference is in the calculation methodology.

Table 6-5 provides a tabulation of the calculation and acceptability of each increment of capital investment considered. Since these three feasible alternatives are a mutually exclusive set of investment alternatives, the base alternative is the one with the least capital investment cost that is economically justified. For Alternative *A*, the PW of the negative cash-flow amounts (at $i = \epsilon\%$)

TABLE 6-5 Comparison of Three Mutually Exclusive Alternatives Using the ERR Method (Example 6-6)

	Alternative Cash Flows			Incremental Analysis of Alternatives		
End of Period	*A*	*B*	*C*	*A*[a]	Δ(*B* – *A*)	Δ(*C* – *A*)
0	–$640,000	–$680,000	–$755,000	–$640,000	–$40,000	–$115,000
1	262,000	–40,000	205,000	262,000	–302,000	–57,000
2	290,000	392,000	406,000	290,000	102,000	116,000
3	302,000	380,000	400,000	302,000	78,000	98,000
4	310,000	380,000	390,000	310,000	70,000	80,000
5	310,000	380,000	390,000	310,000	70,000	80,000
6	260,000	380,000	324,000	260,000	120,000	64,000
		Incremental analysis:				
		Δ PW of negative cash-flow amounts		640,000	291,657	162,498
		Δ FW of positive cash-flow amounts		2,853,535	651,091	685,082
		ERR		28.3%	14.3%	27.1%
		Is increment justified?		Yes	No	Yes

[a] The net cash flow for Alternative *A*, which is the incremental cash flow between making no change ($0) and implementing Alternative *A*.

is just the \$640,000 investment cost. Therefore, the ERR for Alternative A is the following:

$$\$640{,}000(F/P, i'\%, 6) = \$262{,}000(F/P, 20\%, 5) + \cdots + \$260{,}000$$
$$= \$2{,}853{,}535$$
$$(F/P, i'\%, 6) = (1 + i')^6 = \$2{,}853{,}535/\$640{,}000 = 4.4586$$
$$(1 + i') = (4.4586)^{1/6} = 1.2829$$
$$i' = 0.2829, \text{or ERR} = 28.3\%.$$

Using a MARR = 20% per year, this capital investment is justified, and Alternative A is an acceptable base alternative. By using similar calculations, the increment $\Delta(B - A)$, earning 14.3%, is not justified and the increment $\Delta(C - A)$, earning 27.1%, is justified. Therefore, Alternative C is the preferred alternative for the improvement project. Note in this example that revenues varied among the alternatives and that Rule 1, Section 6.2.2, was applied.

By this point in the chapter, three key observations are clear concerning the comparison of mutually exclusive alternatives: (1) equivalent-worth methods are computationally less cumbersome to use, (2) both the equivalent-worth and rate-of-return methods, if used properly, will consistently recommend the best alternative, but (3) rate-of-return methods may not produce correct choices if the analyst or the manager insists on maximizing the rate of return on the total cash flow. That is, incremental investment analysis must be used with rate-of-return methods to ensure that the best alternative is selected.

6.5 Useful Lives Are Unequal among the Alternatives

When the useful lives of mutually exclusive alternatives are unequal, the *repeatability assumption* may be used in their comparison if the study period can be infinite in length or a common multiple of the useful lives. This assumes that the economic estimates for an alternative's initial useful life cycle will be repeated in all subsequent replacement cycles. As we discussed in Section 6.3, this condition is more robust for practical application than it may appear. Another viewpoint is to consider the repeatability assumption as a modeling convenience for the purpose of making a current decision. *When this assumption is applicable to a decision situation, it simplifies comparison of the mutually exclusive alternatives.*

If the repeatability assumption is not applicable to a decision situation, then an appropriate study period needs to be selected (*coterminated assumption*). This is the approach most frequently used in engineering practice because product life cycles are becoming shorter. Often, one or more of the useful lives will be shorter or longer than the selected study period. When this is the case, cash-flow adjustments based on additional assumptions need to be used *so that all the alternatives are compared over the same study period.* The following guidelines apply to this situation:

1. Useful life < Study period
   (a) Cost alternatives: Because each cost alternative has to provide the same level of service over the study period, contracting for the service or leasing the needed equipment for the remaining years may be appropriate. Another potential course of action is to repeat part of the useful life of the original alternative and then use an estimated market value to truncate it at the end of the study period.
   (b) Investment alternatives: The first assumption is that all cash flows will be reinvested in other opportunities available to the firm at the MARR to the end of the study period. A second assumption involves replacing the initial investment with another asset having possibly different cash flows over the remaining life. A convenient solution method is to calculate the FW of each mutually exclusive alternative at the end of the study period. The PW can also be used for investment alternatives, since the FW at the end of the study period, say N, of each alternative is its PW times a common constant $(F/P, i\%, N)$, where $i\% = \text{MARR}$.
2. Useful life > Study period: The most common technique is to truncate the alternative at the end of the study period, using an estimated market value. This assumes that the disposable assets will be sold at the end of the study period at that value.

The underlying principle, as discussed in Section 6.3, is to compare the mutually exclusive alternatives being considered in a decision situation over the same study (analysis) period.

In this section, we explain how to evaluate mutually exclusive alternatives having unequal useful lives. First we consider equivalent-worth methods for making comparisons of alternatives. Then we turn our attention to the use of the rate-of-return method for performing the analysis.

6.5.1 Equivalent-Worth Methods

When the useful lives of alternatives are not the same, the *repeatability assumption* is appropriate if the study period is infinite (very long in length) or a common multiple of the useful lives. Under this assumption, the cash flows for an alternative's initial life cycle will be repeated (i.e., they are identical) in all subsequent replacement cycles. Because this assumption is applicable in many decision situations, it is extremely useful and greatly simplifies the comparison of mutually exclusive alternatives. With repeatability, we will simply compute the AW of each alternative over its own *useful life* and recommend the one having the most economical value (i.e., the alternative with the highest positive AW for investment alternatives and the alternative with the least negative AW for cost alternatives).

Example 6-7 demonstrates the computational advantage of using the AW method (instead of PW or FW) when the repeatability assumption is applicable. Example 6-8 illustrates the use of the coterminated assumption for the same set of alternatives when the selected study period is not a common multiple of the useful lives.

EXAMPLE 6-7

Useful Lives ≠ Study Period: The Repeatability Assumption

The following data have been estimated for two mutually exclusive investment alternatives, A and B, associated with a small engineering project for which revenues as well as expenses are involved. They have useful lives of four and six years, respectively. If MARR = 10% per year, show which alternative is more desirable by using equivalent-worth methods (computed by hand and by spreadsheet). Use the repeatability assumption.

	A	B
Capital investment	\$3,500	\$5,000
Annual cash flow	1,255	1,480
Useful life (years)	4	6
Market value at end of useful life	0	0

Solution

The least common multiple of the useful lives of Alternatives A and B is 12 years. Using the repeatability assumption and a 12-year study period, the first like (identical) replacement of Alternative A would occur at EOY four, and the second would be at EOY eight. For Alternative B, one like replacement would occur at EOY six. This is illustrated in Part 1 of Figure 6-9.

Solution by the PW Method

The PW (or FW) solution must be based on the total study period (12 years). The PW of the initial useful life cycle will be different than the PW of subsequent replacement cycles:

$$\begin{aligned} \mathrm{PW}(10\%)_A &= -\$3{,}500 - \$3{,}500[(P/F, 10\%, 4) + (P/F, 10\%, 8)] \\ &\quad + (\$1{,}255)(P/A, 10\%, 12) \\ &= \$1{,}028, \end{aligned}$$

$$\begin{aligned} \mathrm{PW}(10\%)_B &= -\$5{,}000 - \$5{,}000(P/F, 10\%, 6) \\ &\quad + (\$1{,}480)(P/A, 10\%, 12) \\ &= \$2{,}262. \end{aligned}$$

Based on the PW method, we would select Alternative B.

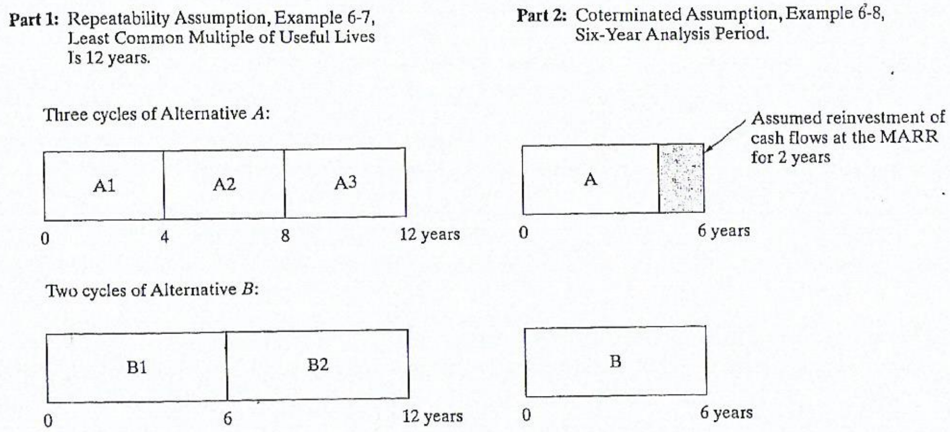


Figure 6-9 Illustration of Repeatability Assumption (Example 6-7) and Coterminated Assumption (Example 6-8)

Solution by the AW Method

The like replacement of assets assumes that the economic estimates for the initial useful life cycle will be repeated in each subsequent replacement cycle. Consequently, the AW will have the same value for each cycle and for the study period (12 years). This is demonstrated in the next AW solution by calculating (1) the AW of each alternative over the 12-year analysis period based on the previous PW values and (2) determining the AW of each alternative over one useful life cycle. Based on the previously calculated PW values, the AW values are

$$\mathrm{AW}(10\%)_A = \$1{,}028(A/P, 10\%, 12) = \$151,$$
$$\mathrm{AW}(10\%)_B = \$2{,}262(A/P, 10\%, 12) = \$332.$$

Next, the AW of each alternative is calculated over one useful life cycle:

$$\mathrm{AW}(10\%)_A = -\$3{,}500(A/P, 10\%, 4) + (\$1{,}255) = \$151,$$
$$\mathrm{AW}(10\%)_B = -\$5{,}000(A/P, 10\%, 6) + (\$1{,}480) = \$332.$$

This confirms that both calculations for each alternative result in the same AW value, and we would again select Alternative B because it has the larger value (\$332).

Spreadsheet Solution

Figure 6-10 shows the spreadsheet solution for this example. EOY cash flows are computed for each alternative over the entire 12-year study period. For Alternative A, the annual cash flow of \$1,255 is combined with the necessary reinvestment cost ($3,500) at the end of each useful life cycle (at EOY 4 and EOY 8). A similar statement can be made for Alternative *B* at the end of its first life cycle (EOY 6 = $1,480 − $5,000). As was the case in the previous solution by hand, Alternative *B* is selected because it has the largest PW (and therefore AW) value. (How much would the annual cash flow for Alternative *A* have to be for it to be as desirable as Alternative *B*? Answer: $1,436.)

	A	B	C
1	MARR =	10%	
2	Study Period =	12	
3			
4		A	B
5	Capital Investment =	$ (3,500)	$ (5,000)
6	Annual cash flow =	$ 1,255	$ 1,480
7	Useful Life =	4	6
8			
9	EOY	A	B
10	0	$ (3,500)	$ (5,000)
11	1	$ 1,255	$ 1,480
12	2	$ 1,255	$ 1,480
13	3	$ 1,255	$ 1,480
14	4	$ (2,245)	$ 1,480
15	5	$ 1,255	$ 1,480
16	6	$ 1,255	$ (3,520)
17	7	$ 1,255	$ 1,480
18	8	$ (2,245)	$ 1,480
19	9	$ 1,255	$ 1,480
20	10	$ 1,255	$ 1,480
21	11	$ 1,255	$ 1,480
22	12	$ 1,255	$ 1,480
23			
24	PW =	$ 1,028	$ 2,262
25	AW =	$151	$332

= B5

= B$6

= B$6 + B$5

= C$6 + C$5

= NPV (B1, B11:B22) + B10

= PMT (B1, B2, − B24)

Figure 6-10 Spreadsheet Solution, Example 6-7

EXAMPLE 6-8 Useful Lives ≠ Study Period: The Coterminated Assumption

Suppose that Example 6-7 is modified such that an analysis period of six years is used (coterminated assumption) instead of 12 years, which was based on repeatability and the least common multiple of the useful lives. Perhaps the responsible manager did not agree with the repeatability assumption and wanted a six-year analysis period because it is the planning horizon used in the company for small investment projects.

Solution

An assumption used for an investment alternative (when useful life is less than the study period) is that all cash flows will be reinvested by the firm at the MARR until the end of the study period. This assumption applies to Alternative *A*, which has a four-year useful life (two years less than the study period), and it is illustrated in Part 2 of Figure 6-9. We use the FW method to analyze this situation:

$$\text{FW}(10\%)_A = [-\$3{,}500(F/P, 10\%, 4) + (\$1{,}255)(F/A, 10\%, 4)](F/P, 10\%, 2)$$
$$= \$847,$$
$$\text{FW}(10\%)_B = -\$5{,}000(F/P, 10\%, 6) + (\$1{,}480)(F/A, 10\%, 6)$$
$$= \$2{,}561.$$

Based on the FW of each alternative at the end of the six-year study period, we would select Alternative *B* because it has the larger value ($2,561).

In the solution to Example 6-7, it was shown that, when repeatability is assumed, the AW of an alternative over a single life cycle is equal to the AW of the alternative over the entire study period. As a result, we can adopt the following rule to simplify the analysis of alternatives with unequal lives when the repeatability assumption is applicable:

> When the repeatability assumption is applied, simply compare the AW amounts of each alternative over its own useful life and select the alternative that maximizes AW.

The capitalized-worth (CW) method was introduced in Chapter 5 as a special variation of the PW method when revenues and expenses occur over an infinite length of time. CW is a convenient basis for comparing mutually exclusive alternatives when the period of needed service is indefinitely long and the repeatability assumption is applicable.

EXAMPLE 6-9 Comparing Alternatives Using CW

We now revisit the problem posed at the beginning of the chapter involving two containment alternatives for coal combustion by-products. Because an indefinitely long study period is specified, we use the CW method to compare the two storage methods. First we compute the AW of each system over its useful life, and then we determine the capitalized worth (refer to Section 5.3.3) over a very long study period.

Wet: $\text{AW}(10\%) = -\$2{,}000{,}000{,}000\ (A/P,\ 10\%,\ 10) - \$300{,}000{,}000$

$= -\$625{,}400{,}000$

$\text{CW}(10\%) = \text{AW}(10\%)/0.10 = -\$6{,}254{,}000{,}000$

Dry: $\text{AW}(10\%) = -\$2{,}500{,}000{,}000\ (A/P,\ 10\%,\ 7) - \$150{,}000{,}000$

$= -\$663{,}500{,}000$

$\text{CW}(10\%) = \text{AW}(10\%)/0.10 = -\$6{,}635{,}000{,}000$

We recommend the wet slurry storage method because it has the lesser negative (greater) CW.

Example 6-10 demonstrates how to deal with situations in which multiple machines are required to satisfy a fixed annual demand for a product or service. Such problems can be solved by using Rule 2 and the repeatability assumption.

EXAMPLE 6-10 AW and Repeatability: Perfect Together!

Three products will be manufactured in a new facility at the Apex Manufacturing Company. They each require an identical manufacturing operation, but different production times, on a broaching machine. Two alternative types of broaching machines (*M1* and *M2*) are being considered for purchase. One machine type must be selected.

For the same level of annual demand for the three products, *annual* production requirements (machine hours) and annual operating expenses (per machine) are listed next. Which machine should be selected if the MARR is 20% per year? Solve by hand and by spreadsheet. Show all work to support your recommendation. (Use Rule 2 on page 253 to make your recommendation.)

Product	Machine *M1*	Machine *M2*
ABC	1,500 hr	900 hr
MNQ	1,750 hr	1,000 hr
STV	2,600 hr	2,300 hr
	5,850 hr	4,200 hr
Capital investment	$15,000 per machine	$22,000 per machine
Expected life	five years	eight years
Annual expenses	$4,000 per machine	$6,000 per machine

Assumptions: The facility will operate 2,000 hours per year. Machine availability is 90% for Machine *M1* and 80% for Machine *M2*. The yield of Machine *M1* is 95%, and the yield of Machine *M2* is 90%. Annual operating expenses are based on an assumed operation of 2,000 hours per year, and workers are paid during any idle time of Machine *M1* or Machine *M2*. Market values of both machines are negligible.

Solution by Hand

The company will need 5,850 hours/[2,000 hours (0.90)(0.95)] = 3.42 (four machines of type *M1*) or 4,200 hours/[2,000 hours (0.80)(0.90)] = 2.92 (three machines of type *M2*). The maximum operation time of 2,000 hours per year

	A	B	C	D	E	F	G
1	MARR =	20%			Facility Operation Hours/year =		2,000
2							
3		M1	M2		Product	M1	M2
4	Capital Investment (ea.)	$ 15,000	$ 22,000		ABC (hrs.)	1,500	900
5	Useful Life	5	8		MNQ (hrs.)	1,750	1,000
6	Annual Expenses (ea.)	$ 4,000	$ 6,000		STV (hrs.)	2,600	2,300
7	Availability	90%	80%				
8	Yield	95%	90%		Total Hrs.	5,850	4,200
9							
10							
11		M1	M2		EOY	M1	M2
12	Annual Available Hours	1,800	1,600		0	$ (60,000)	$ (66,000)
13	Annual Net Yield Hours	1,710	1,440		1	$ (16,000)	$ (18,000)
14	Theoretical Requirement	3.42	2.92		2	$ (16,000)	$ (18,000)
15	Actual Requirement	4	3		3	$ (16,000)	$ (18,000)
16					4	$ (16,000)	$ (18,000)
17					5	$ (16,000)	$ (18,000)
18					6		$ (18,000)
19					7		$ (18,000)
20					8		$ (18,000)
21							
22					AW =	($36,063)	($35,200)

= G1 * B7

= B12 * B8

= F8/B13

= CEILING(B14, 1)

= –B4 * B15

= SUM(F4:F6)

= –B$6 * B$15

= –PMT (B1, B5, NPV (B1, F13:F17) + F12)

= –PMT (B1, C5, NPV (B1, G13:G20) + G12)

Figure 6-11 Spreadsheet Solution, Example 6-10

in the denominator must be multiplied by the availability of each machine and the yield of each machine, as indicated.

The annual cost of ownership, assuming a MARR = 20% per year, is \$15,000(4)(A/P, 20%, 5) = \$20,064 for Machine $M1$ and \$22,000(3)(A/P, 20%, 8) = \$17,200 for Machine $M2$.

There is an excess capacity when four Machine $M1$s and three Machine $M2$s are used to provide the machine-hours (5,850 and 4,200, respectively) just given. If we assume that the operator is paid for idle time he or she may experience on $M1$ or $M2$, the annual expense for the operation of four M1s is 4 machines × \$4,000 per machine = \$16,000. For three $M2$s, the annual expense is 3 machines × \$6,000 per machine = \$18,000.

The total equivalent annual cost for four Machine $M1$s is \$20,064 + \$16,000 = \$36,064. Similarly, the total equivalent annual expense for three Machine $M2$s is \$17,200 + \$18,000 = \$35,200. By a slim margin, Machine $M2$ is the preferred choice to minimize equivalent annual costs with the repeatability assumption.

Spreadsheet Solution

Figure 6-11 on page 279 shows the complete spreadsheet solution for this example. Note the use of the CEILING function in cell B15 to convert the noninteger theoretical number of machines required into an actual requirement. This actual requirement is used to compute the total EOY cash flows for each alternative by multiplying the per-machine costs by the number of machines required.

Since we are assuming repeatability, the EOY cash flows are shown only for the initial useful life span for each alternative. These cash flows are then used to compute the AW of each alternative. As seen previously, purchasing three broaching machines of type $M2$ is preferred to purchasing four machines of type $M1$.

EXAMPLE 6-11

Modeling Estimated Expenses as Arithmetic Gradients

You are a member of an engineering project team that is designing a new processing facility. Your present design task involves the portion of the catalytic system that requires pumping a hydrocarbon slurry that is corrosive and contains abrasive particles. For final analysis and comparison, you have selected two fully lined slurry pump units, of equal output capacity, from different manufacturers. Each unit has a large-diameter impeller required and an integrated electric motor with solid-state controls. Both units will provide the same level of service (support) to the catalytic system but have different useful lives and costs.

	Pump Model	
	SP240	HEPS9
Capital investment	\$33,200	\$47,600
Annual expenses:		
Electrical energy	\$2,165	\$1,720
Maintenance	\$1,100 in year 1, and increasing \$500/yr thereafter	\$500 in year 4, and increasing \$100/yr thereafter
Useful life (years)	5	9
Market value (end of useful life)	0	5,000

The new processing facility is needed by your firm at least as far into the future as the strategic plan forecasts operating requirements. The MARR is 20% per year. Based on this information, which slurry pump should you select?

Solution

Notice that the estimates for maintenance expenses involve an arithmetic gradient series (Chapter 4). A cash-flow diagram is very useful in this situation to help keep track of the various cash-flow series. The cash-flow diagrams for pump models SP240 and HEPS9 are shown in Figure 6-12.

The repeatability assumption is a logical choice for this analysis, and a study period of either infinite or 45 years (least common multiple of the useful lives) in length can be used. With repeatability, the AW over the initial *useful life* of each alternative is the same as its AW over the length of either study period:

$$\begin{aligned}\text{AW}(20\%)_{\text{SP240}} &= -\$33{,}200(A/P, 20\%, 5) - \$2{,}165 \\ &\quad -[\$1{,}100 + \$500(A/G, 20\%, 5)] \\ &= -\$15{,}187,\end{aligned}$$

$$\begin{aligned}\text{AW}(20\%)_{\text{HEPS9}} &= -\$47{,}600(A/P, 20\%, 9) + \$5{,}000(A/F, 20\%, 9) \\ &\quad -\$1{,}720 - [\$500(P/A, 20\%, 6) \\ &\quad +\$100(P/G, 20\%, 6)] \times (P/F, 20\%, 3) \times (A/P, 20\%, 9) \\ &= -\$13{,}622.\end{aligned}$$

Based on Rule 2 (Section 6.2.2), you should select pump model HEPS9, since the AW over its useful life (nine years) has the smaller negative value (−\$13,622).

As additional information, the following two points support in choosing the repeatability assumption in Example 6-11:

1. The repeatability assumption is commensurate with the long planning horizon for the new processing facility and with the design and operating requirements of the catalytic system.

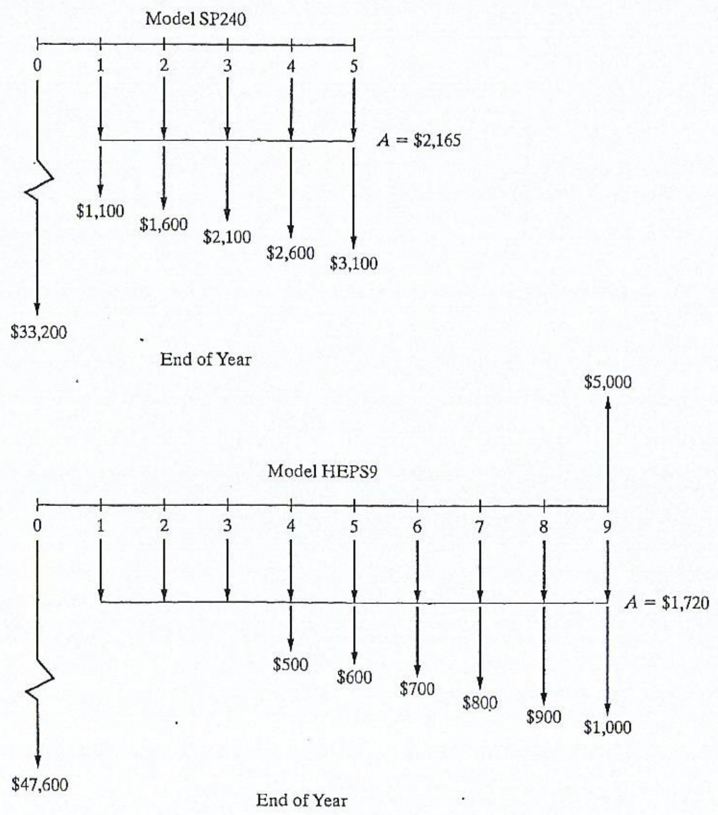


Figure 6-12 Cash-Flow Diagrams for the Pump Models Being Compared in Example 6-11

2. If the initial estimated costs change for future pump-replacement cycles, a logical assumption is that the ratio of the AW values for the two alternatives will remain approximately the same. Competition between the two manufacturers should cause this to happen. Hence, the pump selected (model HEPS9) should continue to be the preferred alternative.

If the existing model is redesigned or new models of slurry pumps become available, however, another study to analyze and compare all feasible alternatives is required before a replacement of the selected pump occurs.

6.5.2 Rate-of-Return Analysis

Up until this point, we have solved problems with unequal lives (see Section 6.5) with the use of the equivalent-worth methods (AW being the most convenient). The analysis of alternatives having unequal lives can also be accomplished by using rate-of-return methods. When the cotermination method is used, the incremental analysis procedure described in Section 6.4 can be applied directly. When the study period is either indefinitely long or equal to a common multiple of the useful lives, however, computing the incremental cash flows can be quite cumbersome. For example, the implicit study period in Example 6-11 is 45 years! In this instance, a more direct approach is useful.

In general, the IRR of an increment of capital is the interest rate, i^*, that equates the equivalent worth of the higher capital investment cost alternative to the equivalent worth of the lower capital investment cost alternative. The decision rule using this approach is that, if $i^* \geq$ MARR, the increment is justified and the alternative with the higher capital investment cost is preferred. So, when repeatability applies, we simply need to develop the AW equation for each alternative over its own useful life and find the interest rate that makes them equal. To demonstrate, consider the alternatives that were analyzed in Example 6-8 by using a MARR of 10% per year and a study period of 12 years (repeatability assumption).

	A	*B*
Capital investment	$3,500	$5,000
Annual cash flow	1,255	1,480
Useful life (years)	4	6

Equating the AW of the alternatives over their respective lives, we get

$$AW_A(i^*\%) = AW_B(i^*\%)$$

$$-\$3{,}500(A/P, i^*\%, 4) + \$1{,}255 = -\$5{,}000(A/P, i^*\%, 6) + \$1{,}480.$$

By trial and error, the IRR of the extra capital needed to repeatedly invest in Alternative B (instead of Alternative A) over the study period is $i^* = 26\%$. Since this value is greater than the MARR, the increment is justified and Alternative B is preferred. This is the same decision arrived at in Example 6-7. The interested student is encouraged to verify that, if the spreadsheet previously displayed in Figure 6-10 were to be expanded to include a column of incremental cash flows over the entire 12-year study period, the IRR of these cash flows, using the IRR function, is indeed 26%.

6.5.3 The Imputed Market Value Technique

Obtaining a current estimate from the marketplace for a piece of equipment or another type of asset is the preferred procedure in engineering practice when a market value at time T < (useful life) is required. This approach, however, may not be feasible in some cases. For example, a type of asset may have low turnover in the marketplace, and information for recent transactions is not available. Hence, it is sometimes necessary to estimate the market value for an asset without current and representative historical data.

The *imputed market value* technique, which is sometimes called the *implied* market value, can be used for this purpose as well as for comparison with marketplace values when current data are available. The estimating procedure used in the technique is based on logical assumptions about the value of the remaining useful life for an asset. If an imputed market value is needed for a piece of equipment, say, at the end of year $T <$ (useful life), the estimate is calculated on the basis of the sum of two parts, as follows:

$$MV_T = [\text{PW at EOY } T \text{ of remaining capital recovery (CR) amounts}]$$
$$+[\text{PW at EOY } T \text{ of original market value at end of useful life}],$$

where PW is computed at $i =$ MARR.

The next example uses information from Example 6-11 to illustrate the technique.

EXAMPLE 6-12 Estimating a New Market Value when Useful Life > Study Period

Use the imputed market value technique to develop an estimated market value for pump model HEPS9 (Example 6-11) at EOY five. The MARR remains 20% per year.

Solution

The original information from Example 6-11 will be used in the solution: capital investment = \$47,600, useful life = nine years, and market value = \$5,000 at the end of useful life.

First, compute the PW at EOY five of the remaining CR amounts [Equation (5-5)]:

$$\text{PW}(20\%)_{\text{CR}} = [\$47{,}600(A/P, 20\%, 9) - \$5{,}000(A/F, 20\%, 9)] \times (P/A, 20\%, 4)$$
$$= \$29{,}949.$$

Next, compute the PW at EOY five of the original MV at the end of useful life (nine years):

$$\text{PW}(20\%)_{\text{MV}} = \$5{,}000(P/F, 20\%, 4) = \$2{,}412.$$

Then, the estimated market value at EOY five ($T = 5$) is as follows:

$$MV_5 = \text{PW}_{\text{CR}} + \text{PW}_{\text{MV}}$$
$$= \$29{,}949 + \$2{,}412 = \$32{,}361.$$

In summary, utilizing the repeatability assumption for unequal lives among alternatives reduces to the simple rule of "comparing alternatives over their useful lives using the AW method, at $i =$ MARR." This simplification, however, may not apply when a study period, selected to be shorter or longer than the common multiple of lives (coterminated assumption), is more appropriate for the decision situation. When utilizing the coterminated assumption, cash flows of alternatives need to be adjusted to terminate at the end of the study period. Adjusting these cash flows usually requires estimating the market value of assets at the end of the study period or extending service to the end of the study period through leasing or some other assumption.

6.6 Personal Finances

Sound financial planning is all about making wise choices for your particular circumstances (e.g., your amount of personal savings, your job security, your attitude toward risk). Thus far in Chapter 6, we have focused on facilitating good decision making from the perspective of a corporation. Now we apply these same principles (remember them from Chapter 1?) to several problems you are likely to face soon in your personal decision making.

Two of the largest investments you'll ever make involve houses and automobiles. This section presents examples of acquiring these assets, usually with borrowed money. Another concern is the extensive use of credit cards (maybe we think that nothing is expensive on a credit card). It turns out that people who use credit cards (almost all of us!) tend to spend more money than others who pay cash or write checks. An enlightening exercise to see how addicted you are to credit cards is to go cold turkey for two months.

A fundamental lesson underlying this section is to save now rather than spending on luxury purchases. By choosing to save now, we are making an attempt to minimize the risk of making poor decisions later on. Check out the savings calculators at `www.Choosetosave.org`.

EXAMPLE 6-13 Automobile Financing Options

You have decided to purchase a new automobile with a hybrid-fueled engine and a six-speed transmission. After the trade-in of your present car, the purchase price of the new automobile is \$30,000. This balance can be financed by an auto dealer at 2.9% APR with payments over 48 months. Alternatively, you can get a \$2,000 discount on the purchase price if you finance the loan balance at an APR of 8.9% over 48 months. Should you accept the 2.9% financing plan or accept the dealer's offer of a \$2,000 rebate with 8.9% financing? Both APRs are compounded monthly.

Solution

In this example, we assume that your objective is to minimize your monthly car payment.

2.9% financing monthly payment:

$$\$30{,}000\ (A/P,\ 2.9\%/12,\ 48\ \text{months}) = \$30{,}000(0.0221) = \$663.00 \text{ per month}$$

8.9% financing monthly payment:

$$\$28{,}000\ (A/P,\ 8.9\%/12, 48\ \text{months}) = \$28{,}000(0.0248) = \$694.90 \text{ per month}$$

Therefore, to minimize your monthly payment, you should select the 2.9% financing option.

What If Questions

When shopping for an automobile you'll find that there are many financing options like the ones in this example available to you. You may find it useful to ask yourself questions such as "how high would the rebate have to be for me to prefer the rebate option," or "how low would the APR have to be for me to select the rebate option?" The answers to these questions can be found through simple equivalence calculations.

(a) How much would the rebate have to be?

Let X = rebate amount. Using the monthly payment of \$663 from the 2.9% financing option, we can solve for the rebate amount that would yield the same monthly payment.

$$(\$30{,}000 - X)(A/P,\ 8.9\%/12 \text{ months},\ 48 \text{ months}) = \$663$$
$$(\$30{,}000 - X)(0.0248) = \$663$$
$$X = \$3{,}266$$

(b) How low would the interest rate have to be?

Now we want to find the interest rate that equates borrowing \$28,000 to 48 monthly payments of \$663. This question is easily solved using a spreadsheet package.

$$\text{RATE}\ (48, -663,\ 28000) = 0.535\% \text{ per month}$$
$$\text{APR} = 0.535\% \times 12 = 6.42\%$$

EXAMPLE 6.14 **Mortgage Financing Options**

A general rule of thumb is that your monthly mortgage payment should not exceed 28% of your household's gross monthly income. Consider the situation of Jerry and Tracy, who just committed to a \$300,000 mortgage on their dream home. They have reduced their financing choices to a 30-year conventional mortgage at 6% APR, or a 30-year interest-only mortgage at 6% APR.

(a) Which mortgage, if either, do they qualify for if their combined gross annual income is \$70,000?

(b) What is the disadvantage in an interest-only mortgage compared to the conventional mortgage?

Solution

(a) Using the general rule of thumb, Jerry and Tracy can afford a monthly mortgage payment of (0.28)(\$70,000/12) = \$1,633. The monthly payment for the conventional mortgage is

$$\$300{,}000\ (A/P,\ 0.5\%,\ 360) = \$1{,}800.$$

For the interest-only mortgage, the monthly payment is

$$(0.005)(\$300{,}000) = \$1{,}500.$$

Thus, the conventional mortgage payment is larger than what the guideline suggests is affordable. This type of loan is marginal because it stretches their budget too much. They easily qualify for the interest-only mortgage because the \$1,500 payment is less than \$1,633.

(b) If home prices fall in the next several years, Jerry and Tracy may have "negative equity" in their home because no principal has been repaid in their monthly interest-only payments. They will not have any buffer to fall back on should they have to sell their house for less than they purchased it for.

It is important to note that interest-only loans don't remain interest-only for the entire loan period. The length of time that interest-only payments may be made is defined in the mortgage contract and can be as short as 5 years or as long as 15 years. After the interest-only period is over, the monthly payment adjusts to include principal and interest. It is calculated to repay the entire loan by the end of the loan period. Suppose Jerry and Tracy's interest-only period was five years. After this time, the monthly payment would become

$$\$300{,}000(A/P,\ 0.5\%,\ 300) = \$1{,}932.90.$$

Before Jerry and Tracy accept this type of loan, they should be confident that they will be able to afford the \$1,932.90 monthly payment in five years.

EXAMPLE 6.15 **Comparison of Two Savings Plans**

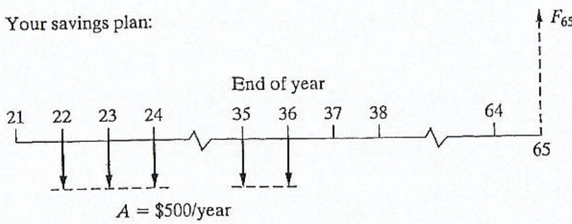
Suppose you start a savings plan in which you save \$500 each year for 15 years. You make your first payment at age 22 and then leave the accumulated sum in the savings plan (and make no more annual payments) until you reach age 65, at which time you withdraw the total accumulated amount. The average annual interest rate you'll earn on this savings plan is 10%.

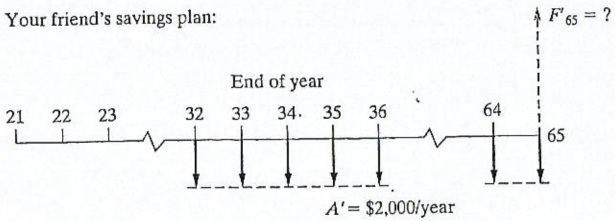
A friend of yours (exactly your age) from Minnesota State University waits 10 years to start her savings plan. (That is, she is 32 years old.) She decides to save \$2,000 each year in an account earning interest at the rate of 10% per year. She will make these annual payments until she is 65 years old, at which time she will withdraw the total accumulated amount.

How old will you be when your friend's *accumulated* savings amount (including interest) exceeds yours? State any assumptions you think are necessary.

Solution

Creating cash-flow diagrams for Example 6-15 is an important first step in solving for the unknown number of years, N, until the future equivalent values of both savings plans are equal. The two diagrams are shown below. The future equivalent (F) of your plan is $\$500(F/A, 10\%, 15)(F/P, 10\%, N-36)$ and that of your friend is $F' = \$2{,}000(F/A, 10\%, N-31)$. It is clear that N, the age at which $F = F'$, is greater than 32. Assuming that the interest rate remains constant at 10% per year, the value of N can be determined by trial and error:





N	Your Plan's F	Friend's F'
36	$15,886	$12,210
38	$19,222	$18,974
39	$21,145	$22,872
40	$23,259	$27,159

By the time you reach age 39, your friend's accumulated savings will exceed yours. (If you had deposited $1,000 instead of $500, you would be over 76 years when your friend's plan surpassed yours. Moral: Start saving early!)

EXAMPLE 6-16 Credit Card Offers

Randy just cancelled his credit card with a large bank. A week later, a representative of the bank called Randy with an offer of a "better" credit card that will advance Randy $2,000 when he accepts it. Randy could not refuse the offer and several days later receives a check for $2,000 from the bank. With this money, Randy decides to buy a new computer.

At the next billing cycle (a month later), the $2,000 advance appears as a charge against Randy's account, and the APR is stated to be 21% (compounded monthly). At this point in time, Randy elects to pay the minimum monthly payment of $40 and cuts up the credit card so that he cannot make any additional purchases.

(a) Over what period of time does this payment extend in order to repay the $2,000 principal?

(b) If Randy decides to repay all remaining principal after having made 15 monthly payments, how much will he repay?

Solution

(a) You may be surprised to know that the majority of credit card companies determine the minimum monthly payment based on a repayment period of 10 years. The monthly interest rate being charged for Randy's card is $21\%/12 = 1.75\%$. We can solve the following equivalence relationship to determine the number of $40 monthly payments required to pay off a loan principal of $2,000.

$$\$2{,}000 = \$40(P/A, 1.75\%, N)$$

This is easily solved using the NPER (rate, payment, principal) function in Excel.

$$\text{NPER}(1.75\%, -40, 2000) = 119.86$$

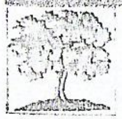
Sure enough, it will take Randy 120 months to pay off this debt.

(b) After having made 15 payments, Randy has 105 payments remaining. To find the single sum payoff for this loan, we simply have to determine the present worth of the remaining payments.

$$\text{Payoff} = \$40(P/A, 1.75\%, 105) = \$1{,}915.96$$

This payoff amount assumes no penalty for early repayment of the loan (which is typically the case when it comes to credit cards). Notice how very little principal ($84.06) was repaid in the early part of the loan.

6.7 CASE STUDY—Ned and Larry's Ice Cream Company



Ned and Larry's Ice Cream Company produces specialty ice cream and frozen yogurt in pint-sized containers. The latest annual performance report praised the firm for its progressive policies but noted that environmental issues like packaging disposal were a concern. In an effort to reduce the effects of consumer disposal of product packaging, the report stated that Ned and Larry's should consider the following proposals:

Proposal *A*—Package all ice cream and frozen yogurt in quart containers.

Proposal *B*—Package all ice cream and frozen yogurt in half-gallon containers.

By packaging the product in containers larger than the current pints, the plastic-coated bleached sulfate board containers will hold more ounces of product per square inch of surface area. The net result is less discarded packaging per ounce of product consumed. Additional advantages to using larger containers include lower packaging costs per ounce and less handling labor per ounce.

Changing to a larger container requires redesign of the packaging and modifications to the filling production line. The existing material-handling equipment can handle the pints and quarts, but additional equipment will be required to handle half-gallons. Any new equipment purchased for proposals *A* and *B* has an expected useful life of six years. The total capital investment for each proposal is shown in the accompanying table. The table summarizes the details of these proposals, as well as the current production of pints.

	Current (Pints)	(*A*) Quarts	(*B*) Half-Gallons
Capital investment	\$0	\$1,200,000	\$1,900,000
Packaging cost per gallon	\$0.256	\$0.225	\$0.210
Handling labor cost per gallon	\$0.128	\$0.120	\$0.119
Postconsumer landfill contribution from discarded packaging (yd^3/yr)	6,500	5,200	4,050

Because Ned and Larry's promotes partnering with suppliers, customers, and the community, they wish to include a portion of the cost to society when evaluating these alternatives. They will consider 50% of the postconsumer landfill cost as part of the costs for each alternative. They have estimated landfill costs to average \$20 per cubic yard nationwide.

Ned and Larry's uses a MARR of 15% per year and IRR analyses to evaluate capital investments. A study period of six years will be used, at which time the equipment purchased for proposals *A* and *B* will have negligible market value. Production will remain constant at 10,625,000 gallons per year. Determine whether Ned and Larry's should package ice cream and frozen yogurt in pints, quarts, or half-gallons.

Solution

Assuming that Ned and Larry's is able to sell all ice cream and frozen yogurt produced, we can focus on the *differences* in costs associated with the three packaging alternatives. Since our recommendation is to be supported by IRR analysis, we must use the *incremental analysis* procedure.

The first step in the incremental analysis procedure is to rank-order the alternatives by increasing capital investment:

$$\text{Current(pints)} \rightarrow A\text{(quarts)} \rightarrow B\text{(half-gallons)}.$$

Now we can compute the incremental difference between producing in quarts (*A*) instead of pints (current) and determine whether the incremental capital investment is justified.

	Δ (*A* − current)
Δ Capital investment	−\$1,200,000
Δ Packaging cost savings	[\$0.031/gal](10,625,000 gal/yr) = \$329,375/yr
Δ Handling cost savings	[\$0.008/gal](10,625,000 gal/yr) = \$85,000/yr
Δ Landfill cost savings	[1,300 yd^3/yr](0.5)(\$20/yd^3) = \$13,000/yr

To determine the IRR of the incremental investment, we can find the interest rate at which the PW of the incremental cash flows is zero.

$$\text{PW}(i'_\Delta\%) = 0 = -\$1{,}200{,}000 + (\$329{,}375 + \$85{,}000 + \$13{,}000)(P/A, i'_\Delta\%, 6).$$

By trial and error, $i'_\Delta\% = 27.2\% >$ MARR. Thus, the investment required to produce the quart-size containers for ice cream is economically justified. Our decision is currently to produce quart-size containers unless the extra investment required to produce in half-gallon-size containers (*B*) earns at least the MARR based on projected savings.

	Δ (*B* − *A*)
Δ Capital investment	−\$700,000
Δ Packaging cost savings	[\$0.015/gal](10,625,000 gal/yr) = \$159,375/yr
Δ Handling cost savings	[\$0.001/gal](10,625,000 gal/yr) = \$10,625/yr
Δ Landfill cost savings	[1,150 yd^3/yr](0.5)(\$20/yd^3) = \$11,500/yr

$$\text{PW}(i'_\Delta\%) = 0 = -\$700{,}000 + (\$159{,}375 + \$10{,}625 + \$11{,}500)(P/A, i'_\Delta\%, 6)$$

By trial and error, $i'_\Delta\% = 14.3\% <$ MARR. Therefore, the extra investment required to produce in half-gallon-size containers (*B*) is not justified by the quantified extra savings.

Based on the preceding incremental IRR analysis, the final recommendation for Ned and Larry's Ice Cream Company is to produce ice cream and frozen yogurt in quart containers (i.e., Proposal *A*). The foregoing incremental analysis procedure indicates that this recommendation will yield an $i'_\Delta\% = 27.2\%$, which is greater than the MARR of 15%. So, by making the recommended change to packaging all ice cream and frozen yogurt in quart containers, not only will Ned and Larry's be delivering great tasting products that are economically attractive to the company, but also the company will be environmentally conscious as well! A win all around! What flavor would you like? ■

6.8 Postevaluation of Results

After comparing alternatives and identifying the preferred course of action (Step 6), we return to the important Step 7 of the economic analysis procedure in Table 1-1. Postaudit reviews, which are appraisals of how a project is progressing against critical milestones, should be conducted during and after a project, recommended by the procedures in Chapter 6, has been approved and funded. This periodic feedback will inform management about the following:

1. Are the planned objectives and milestones being (or have been) attained by the project?
2. Is corrective action required to bring the project in line with the expectations?
3. What can be learned from the selected (and implemented) project that will improve estimating for future projects?

Learning from past decisions (both good and bad!) is critical to improving future decisions.

Periodic postaudit reviews serve to reduce, and often correct for, possible bias in favor of what individual corporate units, divisions, and subsidiaries would like to accomplish to serve their own self-interests. The tendency to estimate optimistically future cash flows and other conditions for pet projects seems to be human nature, but a fair and above-board audit during and after each major project should keep this in check. As a result, estimating project outcomes will be taken more seriously when projects are tracked and monitored over their life cycles. However, a balance must be struck between overly aggressive postaudits and the counter behavior of causing estimators to become overly conservative in their project appraisals.

6.9 Summary

The following chart is a bird's eye view of the methods used in Chapter 6 to identify the preferred alternative.*

When Lives of Alternatives = Study Period

Simplest Approach: Compute the equivalent worth of each alternative (PW, AW, FW, or CW)

Decision Rule: Select alternative with the highest equivalent worth. If comparing cost alternatives, this will be the alternative with the least negative equivalent worth.

Rate-of-Return Methods: The base alternative is the one with the smallest investment cost (if an investment alternative, the $IRR_{base} \geq$ MARR). Use incremental analysis to compare base alternative to next smallest investment cost alternative.

DecisionRule : If $IRR_\Delta \geq$ MARR, eliminate base and move on to next comparison. If $IRR_\Delta <$ MARR, keep base and move on to next comparison.

* Special thanks to Karen Bursic for suggesting this format for the summary.

When Lives of Alternatives Are Different:

Repeatability Assumption:

Simplest Approach: Use AW over each alternative's own life. Select the alternative with the greatest AW.

Rate-of-Return Approach: Set AW of base = AW of next smallest investment cost and solve for the unknown interest rate i_Δ'.

Decision Rule: If $IRR_\Delta \geq$ MARR, eliminate base and move to next comparison. If $IRR_\Delta <$ MARR, keep base and move to next comparison.

Coterminated Assumption:

Simplest Approach: For investment alternatives, the most logical approach is to bring all cash flows to time N, the end of the life of the longest lived alternative without repeating any cash flows (use FW). You could also use the shortest life and the imputed market value for longer lived alternatives.

Rate-of-Return Approach: Perform an incremental analysis with an appropriate study period and assumptions regarding extended or shortened alternative lives.

Problems

The number in parentheses that follows each problem refers to the section from which the problem is taken.

6-1. Three mutually exclusive alternatives are being evaluated, and their costs and revenues are listed in Table P6-1. (6.4)

a. If the MARR is 20% per year and the analysis period is 10 years, use the PW method to determine which alternatives are economically acceptable and which one should be selected.
b. If the total capital investment budget available is $500,000, which alternative should be selected?
c. Which rule (Section 6.2.2) applies? Why?

6-2. A company is analyzing three project proposals. All the projects are mutually exclusive and the budget limit is $10,000,000. Each project's capital investment and annual operating expenses are shown in Table P6-2.

Table for Problem 6-1

	Alternative 1	Alternative 2	Alternative 3
Capital investment	$300,000	$450,000	$600,000
Annual revenues	$200,000	$100,000	$200,000
Annual expenses	$50,000	$50,000	$100,000
Market value	$50,000	$50,000	$100,000
Useful life (years)	10	10	10

Table for Problem 6-2

	1	2	3
Capital investment	$800,000	$600,000	$400,000
Project life	10 years	10 years	10 years
Annual revenue	$450,000	$400,000	$300,000
Annual expenses	$200,000	$180,000	$150,000
Market value	$100,000	$80,000	$60,000

Using the desired MARR of 10% per year, and for the same analysis period, determine which project should be adopted on the basis of the PW method. Confirm your selection using the AW and FW methods. Which rule (Section 6.2.2) applies? Why? (6.4)

6-3. At the Motorola plant in Mesa, Arizona, it is desired to determine whether one-inch-thick insulation or two-inch-thick insulation should be used to reduce heat loss from a long section of steam pipe. The heat loss from this pipe without any insulation would cost $2.00 per linear foot per year. The one-inch insulation will eliminate 88% of the heat loss and will cost $0.60 per foot. Two-inch insulation will eliminate 92% of the heat loss and will cost $1.10 per foot. The steam pipe is 1,000 feet in length and will last for 10 years. MARR = 6% per year. Which insulation thickness should be recommended? (6.4)

6-4. A manufacturing firm is evaluating three automated machines. The estimated production rate and cost data for each machine are given below. The MARR is 15% per year.

	A	B	C
Investment cost	$50,000	$75,000	$60,000
Production rate (per year)	20,000	30,000	25,000
Selling price ($/unit)	$4	$5	$4.5
Variable costs ($/unit)	$2	$3	$1.5
Annual expenses	$20,000	$30,000	$40,000
Market value	$25,000	$25,000	$25,000
Useful life (years)	8	8	8

Annual revenues are based on the number of units sold and the selling price. Annual expenses are based on fixed and variable costs. Determine which selection is preferable based on AW. State your assumptions. (6.4)

6-5. An airport needs a modern material handling system for facilitating access to and from a busy maintenance hanger. A second-hand system will cost $75,000. A new system with improved technology can decrease labor hours by 20% compared to the used system. The new system will cost $150,000 to purchase and install. Both systems have a useful life of five years. The market value of the used system is expected to be $20,000 in five years, and the market value of the new system is anticipated to be $50,000 in five years. Current maintenance activity will require the used system to be operated 8 hours per day for 20 days per month. If labor costs $40 per hour and the MARR is 1% per month, which system should be recommended? (6.4)

6-6. You have been asked to evaluate the economic implications of various methods for cooling condenser effluents from a 200-MW steam-electric plant. In this regard, cooling ponds and once-through cooling systems have been eliminated from consideration because of their adverse ecological effects. It has been decided to use cooling towers to dissipate waste heat to the atmosphere. There are two basic types of cooling towers: wet and dry. Furthermore, heat may be removed from condenser water by (1) forcing (mechanically) air through the tower or (2) allowing heat transfer to occur by making use of natural draft. Consequently, there are four basic cooling tower designs that could be considered. Assuming that the cost of capital to the utility company is 12% per year, your job is to recommend the best alternative (i.e., the least expensive during the service life) in view of the data in Table P6-6. Further, assume that each alternative is capable of satisfactorily removing waste heat from the condensers of a 200-MW power plant.

Alternative Types of Cooling Towers for a 200-Megawatt Fossil-Fired Power Plant Operating at Full Capacity[a] in Problem 6-6

	Alternative			
	Wet Tower Mech. Draft	Wet Tower Natural Draft	Dry Tower Mech. Draft	Dry Tower Natural Draft
Initial cost	$3 million	$8.7 million	$5.1 million	$9.0 million
Power for I.D. fans	40 200-hp induced-draft fans	None	20 200-hp I.D. fans	None
Power for pumps	20 150-hp pumps	20 150-hp pumps	40 100-hp pumps	40 100-hp pumps
Mechanical maintenance/year	$0.15 million	$0.10 million	$0.17 million	$0.12 million
Service life	30 years	30 years	30 years	30 years
Market value	0	0	0	0

[a] 100 hp = 74.6 kW; cost of power to plant is 2.2 cents per kWh or kilowatt-hour; induced-draft fans and pumps operate around the clock for 365 days/year (continuously). Assume that electric motors for pumps and fans are 90% efficient.

What *noneconomic* factors can you identify that might also play a role in the decision-making process? (6.4)

6-7. Three mutually exclusive project alternatives are being evaluated. The estimated cash flows for each alternative are given below. The MARR is 15% per year. At the conclusion of its useful life the investment will be sold. (6.4)

	Project *A*	Project *B*	Project *C*
Investment	$6,000	$8,000	$9,000
Project life	10 years	10 years	10 years
Annual revenue	$5,200	$6,000	$7,500
Annual cost	$2,100	$1,800	$2,000
Salvage value	$1,200	$1,500	$2,500

A decision-maker can select one of these alternatives or decide to select none of them. Make a recommendation using the AW and FW methods.

6-8. Your boss has asked you to evaluate the economics of replacing 1,000 60-Watt incandescent light bulbs (ILBs) with 1,000 compact fluorescent lamps (CFLs) for a particular lighting application. During your investigation you discover that 13-Watt CFLs costing $2.00 each will provide the same illumination as standard 60-Watt ILBs costing $0.50 each. Interestingly, CFLs last, on average, eight times as long as incandescent bulbs. The average life of an ILB is one year over the anticipated usage of 1,000 hours each year. Each incandescent bulb costs $2.00 to install/replace. Installation of a single CFL costs $3.00, and it will also be used 1,000 hours per year.

Electricity costs $0.12 per kiloWatt hour (kWh), and you decide to compare the two lighting options over an 8-year study period. If the MARR is 12% per year, compare the economics of the two alternatives and write a brief report of your findings for the boss. (6.4)

6-9. On her thirty-first birthday, Jean invests $1,000 into her employer's retirement plan, and she continues to make annual $1,000 payments for 10 years. So her total contribution (principal) is $10,000. Jean then stops making payments into her plan and keeps her money in the savings plan untouched for 25 more years. Doug starts putting money aside on his forty-first birthday when he deposits $1,000, and he continues these payments until he gets to be 65 years old. Doug's contributed principal amounts to $25,000 over this period of time.

If Jean's and Doug's retirement plans earn interest of 6% per year, how much will they have accumulated (principal plus interest) when they reach 65 years old? What is the moral of this situation? (6.4)

6-10. A steam generator is needed in the design of a new power plant. This generator can be fired by three different fuels, *A*, *B*, or *C*, with the following cost implications.

	A	B	C
Installed investment cost	$30,000	$55,000	$180,000
Annual fuel expense	X + $7,500	X	X − $1,500
Residual value	None	None	None

If the study period is 20 years and MARR is 8% per year, which type of fuel is most economical? (6.4)

6-11. Acme Semiconductor is expanding its facility and needs to add equipment. There are three process tools under consideration. You have been asked to perform an economic analysis to select the most appropriate tool to acquire. You have gathered the following information for evaluation. Each of these tools has a useful life of seven years. Acme's accounting staff has established a company-wide MARR of 8% per year. Which one of the process tools should be selected? (6.4)

	Tool A	Tool B	Tool C
Investment costs	$55,000	$45,000	$80,000
Annual expenses	$6,250	$8,550	$3,200
Annual revenue	$18,250	$16,750	$20,200
Market value	$18,000	$3,750	$22,000
IRR	11.9%	6.5%	11.0%

6-12. The following cash flow estimates have been developed for two mutually exclusive investment alternatives. (6.4)

EOY	Alternative 1	Alternative 2
0	$10,000	$12,000
1	$4,000	$5,000
2	$2,000	$2,000
3	$2,000	$2,000
4	$2,000	$2,000
5	$7,000	$5,500

a. Find the internal rate of return on the incremental cash flow (A2-A1).

b. Calculate the PW on total investment for each alternative for a MARR of 16% and select the best alternative.

c. Calculate the PW on incremental investment for each alternative for a MARR of 16% and select the best alternative.

6-13. The alternatives for an engineering project to recover most of the energy presently being lost in the primary cooling stage of a chemical processing system have been reduced to three designs. The estimated capital investment amounts and annual expense *savings* are as follows:

	Design		
EOY	ER1	ER2	ER3
0	−$98,600	−$115,000	−$81,200
1	25,800	29,000	19,750
2			
3	$\bar{f} = 6\%$[a]	G = $150[b]	c
4			
5			
6	34,526	29,750	19,750

[a] After year one, the annual savings are estimated to increase at the rate of 6% per year.
[b] After year one, the annual savings are estimated to increase $150 per year.
[c] Uniform sequence of annual savings.

Assume that the MARR is 12% per year, the study period is six years, and the market value is zero for all three designs. Apply an equivalent-worth analysis method to determine the preferred alternative. (6.4)

6-14. Two parcels of land are being considered for a new office building. Both sites cost the same amount but differ mainly in their annual property tax assessments. The parcel in City A has a current property tax of $15,500 per year. This tax is expected to increase by $500 per year starting at EOY 2. The other site, in City B, has a property tax of $12,000 per year with an anticipated increase of $2,000 per year starting at EOY 2. How much money would have to be set aside today for each site to provide for property taxes spanning the next 10 years? The interest rate is 12% per year. (6.4)

6-15. Consider the three mutually exclusive projects that follow. The firm's MARR is 10% per year. (6.4)

EOY	Project 1	Project 2	Project 3
0	−$10,000	−$8,500	−$11,000
1–3	$5,125	$4,450	$5,400

a. Calculate each project's PW.

b. Determine the IRR of each project.

c. Which project would you recommend?

d. Why might one project have the highest PW while a different project has the largest IRR?

6-16. A special-purpose 30-horsepower electric motor has an efficiency of 90%. Its purchase and installation price is $2,200. A second 30-horsepower high-efficiency motor can be purchased for $3,200, and its efficiency is 93%. Either motor will be operated 4,000 hours per year at full load, and electricity costs $0.10 per kilowatt-hour (kWh). MARR = 15% per year, and neither motor will have a market value at the end of the eight-year study period. (6.4)

a. Which motor should be chosen?

b. For an incremental investment of $1,000 in the more efficient motor, what is the PW of the energy savings over the eight-year period?

6-17. Refer to the situation in Problem 6-16. Most motors are operated at a fraction of their rated capacity (i.e., full load) in industrial applications. Suppose the average usage (load factor) of the motor in Problem 6-16 is expected to be 60%. Which motor should be recommended under this condition? (6.4)

6-18. A solar energy research center requires specialized research equipment. For this purpose, four pieces of equipment and their associated cash flows (listed below) are under consideration. One piece of equipment must be selected and the laboratory's MARR is 10% per year. (6.4)

	Equipment			
EOY	1	2	3	4
0	−$150,000	−$80,000	−$125,000	−$100,000
1–3	−$20,000	−$30,000	−$25,000	−$40,000
4	−$30,000	−$30,000	−$25,000	−$45,000
5	−$40,000	−$35,000	−$30,000	−$50,000

a. Use the PW method to rank-order the economic attractiveness of the four pieces of equipment.

b. Determine the interest rate at which the laboratory would be indifferent between Equipments 2 and 3.

6-19. Two mutually exclusive diesel generators are considered for purchase by a power generation company. Information relevant to compare the alternatives is summarized below:

	Generator A	Generator B
Capital investment	$100,000	$80,000
Market value at the end of service life	$35,000	$10,000
Annual fuel and maintenance expenses	$3,000	$5,000
Service life	10 years	10 years

Use the IRR method to determine the better machine to be purchased. The MARR is 10% per year. (6.4)

6-20. Insulated concrete forms (ICF) can be used as a substitute for traditional wood framing in building construction. Heating and cooling bills will be about 50% less than in a similar wood-frame home in Vermont. An ICF home will be approximately 10% more expensive to construct than a wood-frame home.

A typical 2,000 square foot home costs $120 per square foot to build with wood framing in Vermont and costs $200 per month (on average) to heat or cool. What is the IRR on the incremental investment in an equivalent-sized ICF home? The home's residual value with both framing methods in 20 years is expected to be $280,000. (6.4)

6-21. A movie theatre is considering the purchase of a new digital projection system (each showing is as perfect as when it was first filmed). The new ticket price per person would be $7.50, which is $1.00 higher than for the traditional cellulose film projection system. The new projection system would cost $50,000. (6.4)

a. If the theatre expects to sell 20,000 tickets per year, how many years (as an integer) will it take for the theatre to recover the $50,000 investment of the new system (i.e., what is the simple payback period)?

b. What is the discounted payback period (as an integer) if MARR is 20% per year?

c. If the digital system has a life of five years and a salvage value of $5,000 at that time, what is the IRR of the new system?

6-22. Four mutually exclusive projects are being considered for a community gym. The life of equipment in the gym is expected to be 50 years, and the sponsoring agency's MARR is 15% per year. Annual income from the gym has been estimated by a committee and is shown below. Use the IRR method (incrementally) to select the best project. (6.4)

	Alternatives			
	A	B	C	D
Initial investment	$3,000	$2,500	$7,000	$2,700
Annual income	$500	$400	$1,000	$450
ROI	16%	17.5%	10.5%	16%

6-23. In an oil refinery, an equipment is to be installed for the treatment of greenhouse gases and the reduction of emissions. The company requires installing the equipment due to pressure from the government. Two mutually exclusive alternatives have been proposed to rectify the problem:

	Equipment 1	Equipment 2
Capital investment	$125,000	$75,500
Annual operating expenses	$14,000	$5,600

The life of both alternatives is 10 years and the MARR is 10% per year. Use the IRR method (incrementally) to make a recommendation regarding which alternative to select. Can you list some nonmonetary factors that would make you favor Equipment 1? (6.4)

6-24. Are you thinking of bypassing a gasoline-fueled car in favor of a hybrid (gasoline and electric) automobile? Let's take a look at the relative economics of your possible choice. The gasoline-fueled car sells for $20,000 and gets 25 miles per gallon (mpg) of fuel. The alternative hybrid vehicle sells for $25,000 and averages 46 mpg. The resale value of the hybrid car is $2,000 more than that of the gasoline-only car after five years of anticipated ownership. If you drive 15,000 miles per year and gasoline costs $4.00 per gallon, what is the internal rate of return on the incremental investment in the hybrid automobile? (6.4)

6-25. A firm is considering the purchase of a new machine to increase the productivity of existing production process. All the alternatives have a life of 10 years and they have negligible market value after 10 years. Use the IRR method (incrementally) to make your recommendation. The firm's MARR is 10% per year. (6.4)

Alternative	Capital Investment	Annual Operating Cost
1	$100,000	$20,000
2	$110,000	$18,500
3	$125,000	$17,000
4	$130,000	$15,500
5	$150,000	$10,000

6-26. In the Rawhide Company (a leather products distributor), decisions regarding approval of proposals for capital investment are based upon a stipulated MARR of 18% per year. The five packaging devices listed in Table P6-26 were compared, assuming a 10-year life and zero market value for each at that time. Which one (if any) should be selected? Make any additional calculations that you think are needed to make a comparison, using the ERR method. Let $\epsilon = 18\%$. (6.4)

6-27. Refer to Problem 6-2. Solve this problem using the ERR method. Let $\epsilon = 12\%$ per year. (6.4)

6-28. A 50 horsepower motor is required to power a large capacity blower. Two motors A and B, mutually exclusive, have been proposed. Their cost data are given in the next page: (6.5)

Table for Problem 6-26

	Packaging Equipment				
	A	B	C	D	E
Capital investment	$38,000	$50,000	$55,000	$60,000	$70,000
Annual revenues less expenses	11,000	14,100	16,300	16,800	19,200
External rate of return (ERR)	21.1%	20.8%	21.4%	20.7%	20.5%

	Motor A	Motor B
Capital investment	$9,000	$8,000
Annual expenses	$5,000	$6,000
Useful life	10 years	15 years
Market value at the end of useful life	$0	$1,000

The MARR is 5% per year. (6.5)

a. Determine which motor should be selected if the repeatability assumption applies.

b. Determine which alternative should be selected if the analysis period is 15 years, the repeatability assumption does not apply and the machines can be leased for $12,000 per year after the useful life of either machine is over.

6-29. Two electric motors are being considered to drive a centrifugal pump. One of the motors must be selected. Each motor is capable of delivering 60 horsepower (output) to the pumping operation. It is expected that the motors will be in use 800 hours per year. The following data are available: (6.5)

	Motor A	Motor B
Capital investment	$1,200	$1,000
Electrical efficiency	0.90	0.80
Annual maintenance	$160	$100
Useful life	3 years	5 years

a. If electricity costs $0.07 per kilowatt-hour, which motor should be selected if the MARR is 8% per year? Recall that 1 hp = 0.746 kW. Assume repeatability.

b. What is the basic trade-off being made in this problem?

6-30. The National Park Service is considering two plans for rejuvenating the forest and landscape of a large tract of public land. The study period is indefinitely long, and the Park Service's MARR is 10% per year. You have been asked to compare the two plans using the CW method.

The first plan (*Skyline*) calls for an initial investment of $500,000, with expenses of $20,000 per year for the first 20 years and $30,000 per year thereafter. Skyline also requires an expenditure of $200,000 twenty years after the initial investment, and this will repeat every 20 years thereafter. The second plan (*Prairie View*) has an initial investment of $700,000 followed by a single (one-time) investment of $300,000 thirty years later. Prairie View will incur annual expenses of $10,000 forever.

Based on the CW measure, which plan would you recommend? (6.5)

6-31. A new manufacturing facility will produce two products, each of which requires a drilling operation during processing. Two alternative types of drilling machines (D1 and D2) are being considered for purchase. One of these machines must be selected. For the same annual demand, the *annual* production requirements (machine hours) and the annual operating expenses (per machine) are listed in Table P6-31. *Which machine should be selected if the MARR is 15% per year?* Show all your work to support your recommendation. (6.5)

Assumptions: The facility will operate 2,000 hours per year. Machine availability is 80% for Machine D1 and 75% for Machine D2. The yield of D1 is 90%, and the yield of D2 is 80%. Annual operating expenses are based on an assumed operation of 2,000 hours per year, and workers are paid during any idle time of Machine D1 or Machine D2. State any other assumptions needed to solve the problem.

6-32. As the supervisor of a facilities engineering department, you consider mobile cranes to be critical

Table for Problem 6-31

Product	Machine D1	Machine D2
R-43	1,200 hours	800 hours
T-22	2,250 hours	1,550 hours
	3,450 hours	2,350 hours
Capital investment	$16,000/machine	$24,000/machine
Useful life	six years	eight years
Annual expenses	$5,000/machine	$7,500/machine
Market value	$3,000/machine	$4,000/machine

equipment. The purchase of a new medium-sized, truck-mounted crane is being evaluated. The economic estimates for the two best alternatives are shown in the table below. You have selected the longest useful life (nine years) for the study period and would lease a crane for the final three years under Alternative *A*. On the basis of previous experience, the estimated annual leasing cost at that time will be $66,000 per year (plus the annual expenses of $28,800 per year). The MARR is 15% per year. Show that the same selection is made with

(a) the PW method.

(b) the IRR method.

(c) the ERR method.

(d) Would leasing crane A for nine years, assuming the same costs per year as for three years, be preferred over your present selection? (ϵ = MARR = 15%). **(6.4, 6.5)**

	Alternatives	
	A	*B*
Capital investment	$272,000	$346,000
Annual expenses[a]	28,800	19,300
Useful life (years)	6	9
Market value (at end of life)	$25,000	$40,000

[a] Excludes the cost of an operator, which is the same for both alternatives.

6-33. Estimates for a proposed small public facility are as follows: Plan *A* has a first cost of $50,000, a life of 25 years, a $5,000 market value, and annual maintenance expenses of $1,200. Plan *B* has a first cost of $90,000, a life of 50 years, no market value, and annual maintenance expenses of $6,000 for the first 15 years and $1,000 per year for years 16 through 50. Assuming interest at 10% per year, compare the two plans, using the CW method. **(6.4)**

6-34. A manufacturing concern requires a drilling machine to drill an iron sheet of 30mm thickness. To address this need, two mutually exclusive drilling machines are being considered. Doing nothing is not an option. Refer to the data below and state your key assumptions in working out this problem. **(6.5)**

a. Use the PW method to determine which drill should be selected when the MARR is 18% per year.

	Drill *1*	Drill *2*
Capital investment	$2,000	$2,800
Annual revenues	$500	$600
Annual Expenses	$150	$200
Market value at the end of useful life	$400	$0
Useful life	5 years	10 years

b. Which system should be selected when the MARR is 15% per year?

6-35. Your plant must add another boiler to its steam generating system. Bids have been obtained from two boiler manufacturers as follows:

	Boiler *A*	Boiler *B*
Capital investment	$50,000	$120,000
Useful life, *N*	20 years	40 years
Market value at EOY *N*	$10,000	$20,000
Annual operating costs	$9,000	$3,000, increasing $300 per year after the first year

If the MARR is 10% per year, which boiler would you recommend? Use the repeatability assumption. **(6.5)**

6-36. Three mutually exclusive alternatives are being considered for the production equipment at a tissue paper factory. The estimated cash flows for each alternative are given here. (All costs are in thousands.)

	A	*B*	*C*
Capital investment	$2,000	$4,200	$7,000
Annual revenues	3,200	6,000	8,000
Annual costs	2,100	4,000	5,100
Market value at end of useful life	100	420	600
Useful life (years)	5	10	10

Which equipment alternative, if any, should be selected? The firm's MARR is 20% per year. Please state your assumptions. **(6.5)**

6-37. A firm is considering the purchase of one of two new machines. The data on each are given below:

	Machine *A*	Machine *B*
Initial cost	$3,400	$6,500
Service life	3 years	6 years
Market value	$100	$500
Annual expenses	$2,000	$1,800

If *perpetual service* from the machine is assumed, which machine would you recommend? The MARR is 10% per year. **(6.4)**

6-38. Two insulation thickness alternatives have been proposed for a process steam line subject to severe weather conditions. One alternative must be selected. Estimated savings in heat loss and installation cost are given below.

Thickness	Installed Cost	Annual Savings	Life
2 cm	$20,000	$5,000	4 years
5 cm	$40,000	$7,500	6 years

Which thickness would you recommend for a MARR = 15% per year and negligible market (salvage) values? The study period is 12 years. **(6.5)**

6-39. Refer to Problem 6-38. Which thickness would you recommend if the study period was four years? Use the imputed market value technique to estimate the market value of the 5 cm alternative after four years. **(6.5)**

6-40.

a. Compare the probable part cost from Machine *A* and Machine *B*, assuming that each will make the part to the same specification. Which machine yields the lowest part cost? Assume that the MARR = 10% per year. **(6.5)**

b. If the cost of labor can be cut in half by using part-time employees, which machine should be recommended? **(6.5)**

	Machine *A*	Machine *B*
Initial capital investment	$35,000	$150,000
Life	10 years	8 years
Market value	$3,500	$15,000
Parts required per year	10,000	10,000
Labor cost per hour	$16	$20
Time to make one part	20 minutes	10 minutes
Maintenance cost per year	$1,000	$3,000

6-41. Two mutually exclusive electrostatic precipitators (ESP) are being considered as additions to the environmental protection equipment at a power generating company. One of these alternatives must be selected. The estimated cash flows for each alternative are as follows: **(6.5)**

	ESP *A*	ESP *B*
Capital investment	$50,000	$75,000
Annual expenses	$10,000	$6,000
Market value at the end of useful life	$4,000	$12,000
Useful life	5 years	10 years

a. Which electrostatic precipitator alternative should be selected? The firm's MARR is 20% per year. Assume the equipment will be needed indefinitely.

b. Assume the study period shortened to five years. The market value of ESP *B* after 5 years is estimated to be $40,000. Which alternative would you recommend?

6-42. A "greenway" walking trail has been proposed by the city of Richmond. Two mutually exclusive alternative locations for the 2-meter-wide trail have been proposed: one is on flat terrain and is 14 kilometers in length, and the other is in hilly terrain and is 12 kilometers in length.

Planning and site preparation cost is 20% of the asphalt-paving cost, which is $3.00 per square meter. Annual maintenance for the flat terrain trail is 5% of the paving cost, and annual maintenance for the hilly trail is 8% of the paving cost.

If the city's MARR is 6% per year and perpetual life of the trail is assumed, which trail should be recommended to minimize the capitalized cost of this project? **(6.5)**

6-43. Your rich uncle has offered you two living trust payouts for the rest of your life. The positive cash flows of each trust are shown below. If your personal MARR is 10% per year, which trust should you select? (6.5)

End of Year	Trust *A*	Trust *B*
0	$0	$0
1–10	$1,000	$1,200
11–∞	$1,500	$1,300

Note: At 10% per year interest, infinity occurs when $N = 80$ years.

6-44. A firm is considering three mutually exclusive alternatives as a part of the upgradation of an existing manufacturing process. (6.5)

Processes	*C1*	*C2*	*C3*
Installation cost	$200,000	$2,60,000	$2,80,000
Net annual revenue	$76,000	$80,000	$1,00,000
Market value	0	0	0
Useful life	12 years	12 years	6 years
Calculated IRR	25%	22.8%	13.62%

At EOY 6, alternative 3 would be replaced with another alternative 3, having the same installation cost and net annual revenues. If the MARR is 10% per year, which alternative, if any, should be chosen? Use the incremental IRR procedure.

6-45. A piece of production equipment is to be replaced immediately because it no longer meets quality requirements for the end product. The two best alternatives are a used piece of equipment (*E1*) and a new automated model (*E2*). The economic estimates for each are shown in the accompanying table.

	Alternative	
	E1	*E2*
Capital investment	$14,000	$65,000
Annual expenses	$14,000	$9,000
Useful life (years)	5	20
Market value (at end of useful life)	$8,000	$13,000

The MARR is 15% per year.

a. Which alternative is preferred, based on the repeatability assumption? (6.5)

b. Show, for the coterminated assumption with a five-year study period and an imputed market value for Alternative *B*, that the AW of *B* remains the same as it was in Part (a). [And obviously, the selection is the same as in Part (a).] Explain why that occurs in this problem. (6.5)

6-46. A high-pressure pump at a methane gas (biofuel) plant in Memphis costs $30,000 for installation and has an estimated life of 12 years. By the addition of a specialized piece of auxiliary equipment, an annual savings of $400 in operating expense for the pump can be realized, and the estimated life of the pump can be doubled to 24 years. The salvage (market) value of either alternative is negligible at any time. If the MARR is 6% per year, what present expenditure for the auxiliary equipment can plant managers justify spending? (6.5)

6-47. Use the CW method to determine which mutually exclusive bridge design (*L* or *H*) to recommend, based on the data provided in the accompanying table. The MARR is 15% per year. (6.4)

	Bridge Design *L*	Bridge Design *H*
Capital investment	$274,000	$326,000
Annual expenses	$10,000	$8,000
Periodic upgrade cost	$50,000 (every sixth year)	$42,000 (every seventh year)
Market value	0	0
Useful life (years)	83	92

6-48. It has been decided that whole body scanning technology will be used for airport security purposes. Two types of devices are being considered at a certain airport (only one will be selected): back scatter (*B*) and millimeter wave (*W*). The selected scanning equipment is to be used for six years. The economic decision criterion at this airport is to meet or exceed a 15% annual return on its investment. Based on the data found in Table P6-48, which scanner should be selected? Use the PW method of analysis. Which rule, Section 6.2.2, did you use in your analysis? (6.5)

6-49. Two proposals have been offered for streamlining the business operations of a customer call center. Proposal *A* has an investment cost of $30,000, an expected life of five years, property taxes of $450 per year, and no market value. Annual expenses are estimated to be $6,000.

Proposal *B* has an investment cost of $38,000, an expected life of four years, property taxes of $600 per year, and no market value. Its annual operating expenses are expected to be $4,000.

Using a MARR = 10% per year, which proposal should be recommended? Use the AW method and state your assumption(s). (6.5)

Table for Problem 6-48

Factor	*B*	*W*
Capital investment	$210,000	$264,000
Useful life (years)	6	10
Annual expenses	$31,000 in the first year and increasing $2,000 per year thereafter	$19,000 in the first year and increasing 5.7% per year thereafter
Market value (end of useful life)	$21,000	$38,000

6-50. Consider these mutually exclusive alternatives. MARR = 8% per year, so all the alternatives are acceptable. (6.5)

	Alternative		
	A	*B*	*C*
Capital investment (thousands)	$250	$375	$500
Uniform annual savings (thousands)	$40.69	$44.05	$131.90
Useful life (years)	10	20	5
Computed IRR (over useful life)	10%	10%	10%

a. At the end of their useful lives, alternatives *A* and *C* will be replaced with identical replacements (the repeatability assumption) so that a 20-year service requirement (study period) is met. Which alternative should be chosen and why?

b. Now suppose that at the end of their useful lives, alternatives *A* and *C* will be replaced with replacement alternatives having an 8% internal rate of return. Which alternative should be chosen and why? (Note: This assumption allows MEAs to be directly compared with the PW method over their individual useful lives—which violates the repeatability assumption implicit to Chapter 6.)

6-51. The U.S. Border Patrol is considering two remotely operated, solar-powered surveillance systems for a popular border crossing point. One system must be chosen and the study period is indefinitely long. The MARR is 15% per year. Based on the following cash flows, which system should be selected? (6.5)

	System RX-1	System ABY
Capital investment	$24,000	$45,000
Market value at end of useful life	$1,000	$4,000
Annual expenses	$2,500	$1,500
Useful life (years)	5	10

6-52. A firm is considering three mutually exclusive alternatives as part of an upgrade to an existing transportation network.

	I	*II*	*III*
Installed cost	$40,000	$30,000	$20,000
Net annual revenue	$6,400	$5,650	$5,250
Salvage value	0	0	0
Useful life	20 years	20 years	10 years
Calculated IRR	15.0%	18.2%	22.9%

At EOY 10, alternative *III* would be replaced with another alternative *III* having the same installed cost and net annual revenues.

If MARR is 10% per year, which alternative (if any) should be chosen? Use the incremental IRR procedure. (6.5)

6-53. The migration of Asian carp into the Great Lakes must be stopped. Two mutually exclusive alternatives with different useful lives have been proposed to keep the carp from travelling further north:

	Fish Nets	Electric Barriers
Initial investment	$1,000,000	$2,000,000
Annual benefits less expenses	$600,000 (EOY 1)	$1,200,000 (EOY 1)
	$500,000 (EOY 2)	$1,500,000 (EOY 2)
	$500,000 (EOY 3)	$0 (EOY 3)

If the MARR is 12%, which alternative should be chosen if it is specified that the internal rate-of-return method must be utilized? **(6.5)**

6-54. Use the ERR method incrementally to decide whether project *A* or project *B* should be recommended. These are two mutually exclusive cost alternatives, and one of them must be selected. The MARR is 5% per year. Assume repeatability is appropriate for this comparison. **(6.5)**

	Project *A*	Project *B*
Capital investment	$15,000	$25,000
Annual operating expenses	$8,000	$6,500
Useful life	20 years	10 years
Market value	0	0

6-55. Refer to Problem 6-52. Suppose the study period has been coterminated at 10 years. Use the imputed market value technique and determine which alternative is the most economical. **(6.5)**

6-56. You have been requested to offer a recommendation of one of the mutually exclusive industrial sanitation control systems that follow. If the MARR is 15% per year, which system would you select? Use the IRR method. **(6.5)**

	Gravity-fed	Vacuum-led
Capital investment	$24,500	$37,900
Annual receipts less expenses	8,000	8,000
Life in years	5	10
IRR	18.9%	16.5%

6-57. A single-stage centrifugal blower is to be selected for an engineering design application. Suppliers have been consulted, and the choice has been narrowed down to two new models, both made by the same company and both having the same rated capacity and pressure. Both are driven at 3,600 rpm by identical 90-hp electric motors (output).

One blower has a guaranteed efficiency of 72% at full load and is offered installation for $42,000. The other is more expensive because of aerodynamic refinement, which gives it a guaranteed efficiency of 81% at full load.

Except for these differences in efficiency and installed price, the units are equally desirable in other operating characteristics such as durability, maintenance, ease of operation, and quietness. In both cases, plots of efficiency versus amount of air handled are flat in the vicinity of full rated load. The application is such that, whenever the blower is running, it will be at full load.

Assume that both blowers have negligible market values at the end of the useful life, and the firm's MARR is 20% per year. Develop a formula for calculating how much the user could afford to pay for the more efficient unit. (*Hint:* You need to specify important parameters and use them in your formula, and remember 1 hp = 0.746 kW.) **(6.4)**

6-58. Refer to the data given in Problem 6-53. The EOY 3 estimate for the "Fish Nets" alternative is thought to be overly optimistic. A more realistic estimate is $400,000. If the MARR is 20%, what is the recommendation now? **(6.5)**

6-59. A high-quality filter for tap water and a reusable drinking container cost $75 and typically will last for three years. If a family drinks an average of eight cups of water per day (one cup is 8 fluid ounces), the consumption of water is 547.5 gallons over a three-year period. Tap water costs $0.01 per gallon. Alternatively, bottled water may be purchased in 16-ounce bottles for $1.29 each. Compare the present worth of tap filtered water versus bottled water if the family can earn 10% per year on their investments. Assume uniform water consumption over the three-year study period. **(6.6)**

6-60. Your FICO score makes a big difference in how lenders determine what interest rate to charge you. Consider the situation faced by Edward and Jorge. Edward has a fairly poor FICO score of 660 and, as a result, pays 18.0% APR on the unpaid balance of his credit card. Jorge has a FICO score of 740 and pays only 7.3% APR on the unpaid balance of his credit card. If both persons carry an average balance of $3,000 on their credit cards for three years, how much more money will Edward repay compared with what Jorge owes (moral: you want a high FICO score)? Assume monthly compounding of interest. **(6.6)**

6-61. Automobile repair shops typically remind customers to change their oil and oil filter every 3,000 miles. Let's call this Strategy *1*. Usually you can save money by following another legitimate guideline—your automobile user's manual. It suggests changing your oil every 5,000–7,000 miles. We'll call this Strategy 2.

To conserve a nonrenewable and valuable resource, you decide to follow Strategy 2 and change your oil every 5,000 miles (you are conservative). You drive an average of 15,000 miles per year, and you expect each oil change to cost $30. Usually you drive your car 90,000 miles before trading it in on a new one. **(6.6)**

a. Develop an EOY cash-flow diagram for each strategy.

b. If your personal MARR is 10% per year, how much will you save with Strategy 2 over Strategy 1, expressed as a lump sum at the present time?

6-62. On your student loans, if possible, try to make interest-only payments while you are still in school. If interest is not repaid, it folds into principal after graduation and can cost you hundreds (or thousands) of extra dollars in finance charges.

For example, Sara borrowed $5,000 at the beginning of her freshman year and another $5,000 at the beginning of her junior year. The interest rate (APR) is 9% per year, compounded monthly, so Sara's interest accumulates at 0.75% per month. Sara will repay what she owes as an ordinary annuity over 60 months, starting one month after she graduates in the summer term of her fourth full year of college. **(6.6)**

a. How much money does Sara owe upon graduation if she pays off monthly interest during school?

b. How much money does Sara owe if she pays no interest at all during her school years?

c. After graduation, what is the amount of the monthly loan repayment in Parts (a) and (b)?

d. How much interest does Sara repay without interest payments during school and with interest payments while in college?

6-63. A 30-year fixed rate–mortgage is available now at 7% APR. Monthly payments (360 of them) will be made on a $200,000 loan. The realtor says that if you wait several months to obtain a mortgage, the APR could be as high as 8%, "and the monthly payments will jump by 14%." Is the realtor's claim true? If not, what percent increase in monthly payment will result from this 1% increase in APR? **(6.6)**

6-64. A certain U.S. government savings bond can be purchased for $7,500. This bond will be worth $10,000 when it matures in five years. As an alternative, a 60-month certificate of deposit (CD) can be purchased for $7,500 from a local bank, and the CD yields 6.25% per year. Which is the better investment if your personal MARR is 5% per year? **(6.6)**

6-65. Your university advertises that next year's "discounted" tuition of $8,000 can be paid in full by July 1. The other option is to pay the first semester's tuition of $4,200 by July 1, with the remaining $4,200 due on January 1. Assuming you and/or your parents have $8,000 available for the full year's tuition, which tuition plan is more economical if you (or your parents) can earn 6% annually on a six-month CD? **(6.6)**

6-66. Bob and Sally have just bought a new house and they are considering the installation of 10 compact fluorescent bulb (CFB) fixtures instead of the 10 conventional incandescent lighting fixtures (which cost a total of $500) typically installed by the builder. According to the home builder, CFBs use 70% less electricity than incandescent bulbs, and they last 10 times longer before the bulb needs replacement. Bob calculates that 1,000 watts of incandescent lighting will be required in the house for 3,000 hours of usage per year. The cost of electricity is $0.10 per kWh. Also, each CFB will save 150 pounds of CO_2 per year. Each pound of CO_2 has a penalty of $0.02. If incandescent bulbs cost $0.75 each and last for one year, use a 10-year study period to determine the maximum cost of CFB fixtures and bulbs that can be justified in this house. MARR = 8% per year. **(6.4)**

6-67. Three models of baseball bats will be manufactured in a new plant in Pulaski. Each bat requires some manufacturing time at either Lathe *1* or Lathe 2, according to the following table. Your task is to help decide which type of lathe to install. Show and explain all work to support your recommendation. **(6.5)**

Machining Hours for the Production of Baseball Bats		
Product	Lathe 1 (L1)	Lathe 2 (L2)
Wood bat	1,600 hr	950 hr
Aluminum bat	1,800 hr	1,100 hr
Kevlar bat	2,750 hr	2,350 hr
Total machining hours	6,150 hr	4,400 hr

The plant will operate 3,000 hours per year. Machine availability is 85% for Lathe 1 and 90% for Lathe 2. Scrap rates for the two lathes are 5% versus 10% for L1 and L2, respectively. Cash flows and expected lives for the two lathes are given in the adjacent table.

Annual operating expenses are based on an assumed operation of 3,000 hours per year, and workers are paid during any idle time of L1 and L2. Upper management has decided that MARR = 18% per year.

a. How many type L1 lathes will be required to meet the machine-hour requirement?
b. What is the CR cost of the required type L2 lathes?
c. What is the annual operating expense of the type L2 lathes?
d. Which type of lathe should be selected on the basis of lowest *total* equivalent annual cost?

Cash Flows and Expected Lives for L1 and L2

	Lathe 1 (L1)	Lathe 2 (L2)
Capital investment	$18,000 per lathe	$25,000 per lathe
Expected life	7 years	11 years
Annual expenses	$5,000 per lathe	$9,500 per lathe

Spreadsheet Exercises

6-68. Refer to Example 6-3. Re-evaluate the recommended alternative if (a) the MARR = 15% per year; (b) the selling price is $0.50 per good unit; and (c) rejected units can be sold as scrap for $0.10 per unit. Evaluate each change individually. (d) What is the recommended alternative if all three of these changes occur simultaneously? **(6.4)**

6-69. Wilbur is a college student who desires to establish a long-term Roth IRA account with $4,000 that his grandmother gifted to him. He intends to invest the money in a mutual fund that earns an expected 5% per year on his account (this is a very conservative estimate). The sales agent says, "There is a 5% up-front commission (payable now) on a Class A account, which every year thereafter then charges a 0.61% management fee. A Class B account has no up-front commission, but its management fee is 2.35% in year one, 0.34% in year two, and 1.37% per year thereafter." Set up a spreadsheet to determine how many years are required before the worth of the Class A account overtakes (is preferred to) the Class B account. **(6.6)**

6-70. Create a spreadsheet to solve Example 6-11. What would the capital investment amount for pump SP240 have to be such that the firm would be indifferent as to which pump model is selected? **(6.5)**

6-71. Using the data from Problem 6-7, create a spreadsheet to identify the best design using an IRR analysis. **(6.4)**

Case Study Exercises

6-72. What other factors might you include in such an analysis? Should projected revenues be included in the analysis? **(6.7)**

6-73. Would the recommendation change if landfill costs double to $40 per cubic yard? **(6.7)**

6-74. Go to your local grocery store to obtain the price of ice cream packaged in different-sized containers. Use this information (along with additional assumptions you feel necessary) to make a recommendation. **(6.7)**

FE Practice Problems

6-75. You can earn thousands of dollars more from your mutual funds. The secret is to purchase no-load (no front-end commission) funds directly from companies specializing in this area. You can save 5.75% commission compared to "loaded" mutual funds when you buy no-load funds. That means you earn $4,000 more on a $23,000 investment over 10 years. In this situation, the average annual interest rate of the fund is most closely which answer below? **(6.6)**

(a) 6% (b) 12% (c) 15%
(d) 9%

6-76. The Tree Top Airline (TTA) is a small feeder-freight line started with very limited capital to serve the independent petroleum operators in the arid Southwest. All of its planes are identical, although they are painted different colors. TTA has been contracting its overhaul work to Alamo Airmotive for $40,000 per plane per year. TTA estimates that, by building a $500,000 maintenance facility with a life of 15 years and a residual (market) value of $100,000 at the end of its life, they could handle their own overhaul at a cost of only $30,000 per plane per year. What is the minimum number of planes they must operate to make it economically feasible to build this facility? The MARR is 10% per year. **(6.4)**

(a) 7 (b) 4 (c) 5
(d) 3 (e) 8

6-77. Complete the following analysis of investment alternatives and select the preferred alternative. The study period is three years and the MARR = 15% per year. **(6.4)**

	Alternative A	Alternative B	Alternative C
Capital investment	$11,000	$20,000	$13,000
Annual revenues	4,000	8,000	5,540
Annual costs	250	400	400
Market value at EOY 3	5,000	5,500	2,800
PW(15%)	850	???	577

(a) Alternative A (b) Alternative B
(c) Alternative C (d) none of these

6-78. Complete the following analysis of cost alternatives and select the preferred alternative. The study period is 10 years and the MARR = 12% per year. "Do Nothing" is not an option. **(6.4)**

	A	B	C	D
Capital investment	$15,000	$16,000	$15,000	$18,000
Annual costs	250	300	400	100
Market value at EOY 10	1,000	1,300	1,850	2,000
FW(12%)	−$49,975	−$53,658	???	−$55,660

(a) Alternative A (b) Alternative B
(c) Alternative C (d) Alternative D

6-79. The Ford Motor Company is considering three mutually exclusive electronic stability control systems for protection against rollover of its automobiles. The investment period is four years (equal lives), and the MARR is 12% per year. Data for fixturing costs of the systems are given below.

Alternative	IRR	Capital Investment	Annual Receipts Less Expenses	Salvage Value
A	19.2%	$12,000	$4,000	$3,000
B	18%	$15,800	$5,200	$3,500
C	23%	$8,000	$3,000	$1,500

Which alternative should the company select? **(6.4)**

(a) Alternative A (b) Alternative B
(c) Alternative C (d) Do nothing

6-80. For the following table, assume a MARR of 10% per year and a useful life for each alternative of six years that equals the study period. The rank-order of alternatives from least capital investment to greatest capital investment is Do Nothing → A → C → B. Complete the IRR analysis by selecting the preferred alternative. **(6.4)**

	Do Nothing → A	A → C	C → B
Δ Capital investment	−$15,000	−$2,000	−$3,000
Δ Annual revenues	4,000	900	460
Δ Annual costs	−1,000	−150	100
Δ Market value	6,000	−2,220	3,350
Δ IRR	12.7%	10.9%	???

(a) Do nothing (b) Alternative *A*
(c) Alternative *B* (d) Alternative C

6-81. For the following table, assume a MARR of 15% per year and a useful life for each alternative is eight years which equals the study period. The rank-order of alternatives from least capital investment to greatest capital investment is *Z* → *Y* → *W* → *X*. Complete the incremental analysis by selecting the preferred alternative. "Do nothing" is not an option. **(6.4)**

	Z → *Y*	*Y* → *W*	*W* → *X*
Capital investment	−$250	−$400	−$650
Annual cost savings	70	90	25
Market value	100	50	150
PW(15%)	97	20	???

(a) Alternative *W* (b) Alternative *X*
(c) Alternative *Y* (d) Alternative *Z*

6-82. After the base alternative has been identified, the first comparison to be made in an incremental analysis should be which of the following?

(a) C → *B* (b) *A* → *B* (c) *D* → *E*
(d) C → *D* (e) *D* → C

6-83. Using a MARR of 15%, the PW of the investment in *A* when compared *incrementally* to C is most nearly:

(a) −$69,000 (b) −$21,000 (c) $20,000
(d) $61,000 (e) $53,000

6-84. The IRR for Alternative *A* is most nearly:

(a) 30% (b) 15% (c) 36%
(d) 10% (e) 20%

6-85. Using a MARR of 15%, the preferred Alternative is:

(a) Do nothing (b) Alt. *A* (c) Alt. *B*
(d) Alt. *C* (e) Alt. *D* (f) Alt. *E*

6-86. Consider the mutually exclusive alternatives given in the table on p. 309. The MARR is 10% per year. Assuming repeatability, which alternative should the company select? **(6.5)**

Data for Problems 6-82 through 6-85

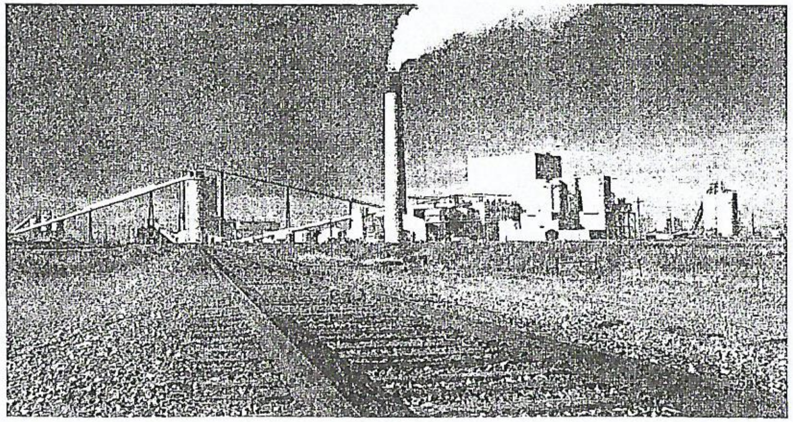
	A	*B*	*C*	*D*	*E*
Capital investment	$60,000	$90,000	$40,000	$30,000	$70,000
Annual expenses	30,000	40,000	25,000	15,000	35,000
Annual revenues	50,000	52,000	38,000	28,000	45,000
Market value at EOY 10	10,000	15,000	10,000	10,000	15,000
IRR	???	7.4%	30.8%	42.5%	9.2%

	Alternative		
	X	*Y*	*Z*
Capital investment (thousands)	$800,000	$350,000	$400,000
Uniform annual savings (thousands)	$131,900	$40,690	$44,050
Useful life (years)	5	10	20

(a) Alternative *Z* (b) Alternative *X*
(c) Alternative *Y* (d) Do nothing

CHAPTER 7

Depreciation and Income Taxes



The objective of Chapter 7 is to explain how depreciation affects income taxes and how income taxes affect economic decision making.

The After-Tax Cost of Producing Electricity

The PennCo Electric Utility Company is going to construct an 800 megawatt (MW) coal-fired plant that will have a 30-year operating life. The $1.12-billion construction cost will be depreciated with the straight-line method over 30 years to a terminal book value of zero. Its market value will also be negligible after 30 years. This plant's efficiency will be 35%, its capacity factor is estimated to be 80%, and annual operating and maintenance expenses are expected to be $0.02 per kilowatt hour (kWh). In addition, the cost of coal is estimated to average $3.50 per million Btu, and the tax on carbon dioxide emissions will be $15 per metric ton. Burning coal emits 90 metric tons of CO_2 per billion Btu produced. If the effective income tax rate for PennCo is 40%, what is the after-tax cost of electricity per year for this plant? The firm's after-tax minimum attractive rate of return (MARR) is 10% per year. After studying this chapter, you will be able to include the impact of income taxes on engineering projects and answer this question (see Example 7-14).

Nothing in this world is certain but death and taxes.

—Benjamin Franklin (1789)

7.1 Introduction

Taxes have been collected since the dawn of civilization. Interestingly, in the United States, a federal income tax did not exist until March 13, 1913, when Congress enacted the Sixteenth Amendment to the Constitution.* Most organizations consider the effect of income taxes on the financial results of a proposed engineering project because income taxes usually represent a significant cash outflow that cannot be ignored in decision making. Based on the previous chapters, we can now describe how income tax liabilities (or credits) and after-tax cash flows are determined in engineering practice. In this chapter, an *After-Tax Cash Flow (ATCF) procedure* will be used in capital investment analysis because it avoids innumerable problems associated with measuring corporate net income. This theoretically sound procedure provides a quick and relatively easy way to define a project's profitability.†

Because the amount of material included in the Internal Revenue Code (and in state and municipal laws, where they exist) is very extensive, only selected parts of the subject can be discussed within the scope of this textbook. Our focus in this chapter is *federal corporate income taxes* and their general effect on the financial results of proposed engineering projects. The material presented is for educational purposes. In practice, you should seek expert counsel when analyzing a specific project.

Due to the effect of *depreciation* on the ATCFs of a project, this topic is discussed first. The selected material on depreciation will then be used in the remainder of the chapter for accomplishing after-tax analyses of engineering projects.

7.2 Depreciation Concepts and Terminology

Depreciation is the decrease in value of physical properties with the passage of time and use. More specifically, depreciation is an *accounting concept* that establishes an annual deduction against before-tax income such that the effect of time and use on an asset's value can be reflected in a firm's financial statements.

* During the Civil War, a federal income tax rate of 3% was initially imposed in 1862 by the Commissioner of Internal Revenue to help pay for war expenditures. The federal rate was later raised to 10%, but eventually eliminated in 1872.

† ATCF is used throughout this book as a measure of the after-tax profitability of an investment project. Other measures, however, are employed for this purpose by management accountants (net income after taxes) and finance professionals (free cash flow). For additional information, refer to Horngren, Sundem, and Stratton (*Management Accounting*) and Bodie and Merton (*Finance*) in Appendix F.

Depreciation is a noncash cost that is intended to "match" the yearly fraction of value used by an asset in the production of income over the asset's life. The actual amount of depreciation can never be established until the asset is retired from service. Because depreciation is a *noncash cost* that affects income taxes, we must consider it properly when making after-tax engineering economy studies.

Depreciable property is property for which depreciation is allowed under federal, state, or municipal income tax laws and regulations. To determine whether depreciation deductions can be taken, the classification of various types of property must be understood. In general, property is depreciable if it meets the following basic requirements:

1. It must be used in business or held to produce income.
2. It must have a determinable useful life (defined in Section 7.2.2), and the life must be longer than one year.
3. It must be something that wears out, decays, gets used up, becomes obsolete, or loses value from natural causes.
4. It is not inventory, stock in trade, or investment property.

Depreciable property is classified as either *tangible* or *intangible*. Tangible property can be seen or touched, and it includes two main types called *personal property* and *real property*. Personal property includes assets such as machinery, vehicles, equipment, furniture, and similar items. In contrast, real property is land and generally anything that is erected on, growing on, or attached to land. Land itself, however, is not depreciable, because it does not have a determinable life.

Intangible property is personal property such as a copyright, patent, or franchise. We will not discuss the depreciation of intangible assets in this chapter because engineering projects rarely include this class of property.

A company can begin to depreciate property it owns when the property is *placed in service* for use in the business and for the production of income. Property is considered to be placed in service when it is ready and available for a specific use, even if it is not actually used yet. Depreciation stops when the cost of placing an asset in service has been recovered or when the asset is sold, whichever occurs first.

7.2.1 Depreciation Methods and Related Time Periods

The depreciation methods permitted under the Internal Revenue Code have changed with time. In general, the following summary indicates the *primary methods used for property placed in service during three distinct time periods:*

Before 1981 Several methods could be elected for depreciating property placed in service before 1981. The primary methods used were Straight-Line (SL), Declining-Balance (DB), and Sum of the Years Digits (SYD). We will refer to these methods, collectively, as the *classical*, or *historical*, *methods* of depreciation.

After 1980 and before 1987 For federal income taxes, tangible property placed in service during this period must be depreciated by using the Accelerated Cost Recovery System (ACRS). This system was implemented by the Economic Recovery Tax Act of 1981 (ERTA).

After 1986 The Tax Reform Act of 1986 (TRA86) was one of the most extensive income tax reforms in the history of the United States. This act modified the previous ACRS implemented under ERTA and requires the use of the Modified Accelerated Cost Recovery System (MACRS) for the depreciation of tangible property placed in service after 1986.

A description of the selected (historical) methods of depreciation is included in the chapter for several important reasons. They apply directly to property placed in service prior to 1981, as well as for property, such as intangible property (which requires the SL method), that does not qualify for ACRS or MACRS. Also, these methods are often specified by the tax laws and regulations of state and municipal governments in the United States and are used for depreciation purposes in other countries. In addition, as we shall see in Section 7.4, the DB and the SL methods are used in determining the annual recovery rates under MACRS.

We do not discuss the application of ACRS in the chapter, but readily available Internal Revenue Service (IRS) publications describe its use.* Selected parts of the MACRS, however, are described and illustrated because this system applies to depreciable property in present and future engineering projects.

7.2.2 Additional Definitions

Because this chapter uses many terms that are not generally included in the vocabulary of engineering education and practice, an abbreviated set of definitions is presented here. The following list is intended to supplement the previous definitions provided in this section:

Adjusted (cost) basis The original cost basis of the asset, adjusted by allowable increases or decreases, is used to compute depreciation deductions. For example, the cost of any improvement to a capital asset with a useful life *greater* than one year increases the original cost basis, and a casualty or theft loss decreases it. If the basis is altered, the depreciation deduction may need to be adjusted.

Basis or cost basis The initial cost of acquiring an asset (purchase price plus any sales taxes), including transportation expenses and other normal costs of making the asset serviceable for its intended use; this amount is also called the *unadjusted cost basis*.

Book value (BV) The worth of a depreciable property as shown on the accounting records of a company. It is the original cost basis of the property, including any adjustments, less all allowable depreciation deductions. It thus represents the amount of capital that remains invested in the property and must be recovered in the future through the accounting process. The BV of a property may not be an accurate measure of its market value. In general, the BV of a property at the end of year k is

$$(\text{Book value})_k = \text{adjusted cost basis} - \sum_{j=1}^{k} (\text{depreciation deduction})_j. \quad (7\text{-}1)$$

* Useful references on material in this chapter, available from the Internal Revenue Service in an annually updated version, are Publication 534 (*Depreciation*), Publication 334 (*Tax Guide for Small Business*), Publication 542 (*Tax Information on Corporations*), and Publication 544 (*Sales and Other Dispositions of Assets*).

Market value (MV) The amount that will be paid by a willing buyer to a willing seller for a property, where each has equal advantage and is under no compulsion to buy or sell. The MV approximates the present value of what will be received through ownership of the property, including the time value of money (or profit).

Recovery period The number of years over which the basis of a property is recovered through the accounting process. For the classical methods of depreciation, this period is normally the useful life. Under MACRS, this period is the *property class* for the General Depreciation System (GDS), and it is the *class life* for the Alternative Depreciation System (ADS) (see Section 7.4).

Recovery rate A percentage (expressed in decimal form) for each year of the MACRS recovery period that is utilized to compute an annual depreciation deduction.

Salvage value (SV) The estimated value of a property at the end of its useful life.* It is the expected selling price of a property when the asset can no longer be used productively by its owner. The term *net salvage value* is used when the owner will incur expenses in disposing of the property, and these cash outflows must be deducted from the cash inflows to obtain a final net SV. Under MACRS, the SV of a depreciable property is defined to be zero.

Useful life The expected (estimated) period that a property will be used in a trade or business to produce income. It is not how long the property will last but how long the owner expects to productively use it.

7.3 The Classical (Historical) Depreciation Methods

This section describes and illustrates the SL and DB methods of calculating depreciation deductions. As mentioned in Section 7.2, these historical methods continue to apply, directly and indirectly, to the depreciation of property. Also included is a discussion of the units of production method.

7.3.1 Straight-Line (SL) Method

SL depreciation is the simplest depreciation method. It assumes that a constant amount is depreciated each year over the depreciable (useful) life of the asset. If we define

N = depreciable life of the asset in years;

B = cost basis, including allowable adjustments;

d_k = annual depreciation deduction in year $k (1 \leq k \leq N)$;

BV_k = book value at end of year k;

SV_N = estimated salvage value at end of year N; and

d_k^* = cumulative depreciation through year k,

* We often use the term market value (MV) in place of salvage value (SV).

then
$$d_k = (B - SV_N)/N, \quad (7\text{-}2)$$
$$d_k^* = k \cdot d_k \text{ for } 1 \leq k \leq N, \quad (7\text{-}3)$$
$$BV_k = B - d_k^*. \quad (7\text{-}4)$$

Note that, for this method, you must have an estimate of the final SV, which will also be the final BV at the end of year N. In some cases, the estimated SV_N may not equal an asset's actual terminal MV.

EXAMPLE 7-1

SL Depreciation

A laser surgical tool has a cost basis of $200,000 and a five-year depreciable life. The estimated SV of the laser is $20,000 at the end of five years. Determine the annual depreciation amounts using the SL method. Tabulate the annual depreciation amounts and the book value of the laser at the end of each year.

Solution

The depreciation amount, cumulative depreciation, and BV for each year are obtained by applying Equations (7-2), (7-3), and (7-4). Sample calculations for year three are as follows:

$$d_3 = \frac{\$200{,}000 - \$20{,}000}{5} = \$36{,}000$$

$$d_3^* = 3\left(\frac{\$200{,}000 - \$20{,}000}{5}\right) = \$108{,}000$$

$$BV_3 = \$200{,}000 - \$108{,}000 = \$92{,}000$$

The depreciation and BV amounts for each year are shown in the following table.

EOY, k	d_k	BV_k
0	—	$200,000
1	$36,000	$164,000
2	$36,000	$128,000
3	$36,000	$92,000
4	$36,000	$56,000
5	$36,000	$20,000

Note that the BV at the end of the depreciable life is equal to the SV used to calculate the yearly depreciation amount.

7.3.2 Declining-Balance (DB) Method

In the DB method, sometimes called the *constant-percentage method* or the *Matheson formula*, it is assumed that the annual cost of depreciation is a fixed percentage of the BV at the *beginning* of the year. The ratio of the depreciation in any one year to the BV at the beginning of the year is constant throughout the life of the asset and is designated by R ($0 \leq R \leq 1$). In this method, $R = 2/N$ when a 200% DB is being used (i.e., twice the SL rate of $1/N$), and N equals the depreciable (useful) life of an asset. If the 150% DB method is specified, then $R = 1.5/N$. The following relationships hold true for the DB method:

$$d_1 = B(R), \quad (7\text{-}5)$$

$$d_k = B(1-R)^{k-1}(R), \quad (7\text{-}6)$$

$$d_k^* = B[1-(1-R)^k], \quad (7\text{-}7)$$

$$BV_k = B(1-R)^k. \quad (7\text{-}8)$$

Notice that Equations (7-5) through (7-8) do not contain a term for SV_N.

EXAMPLE 7-2 **DB Depreciation**

A new electric saw for cutting small pieces of lumber in a furniture manufacturing plant has a cost basis of $4,000 and a 10-year depreciable life. The estimated SV of the saw is zero at the end of 10 years. Use the DB method to calculate the annual depreciation amounts when

(a) $R = 2/N$ (200% DB method)
(b) $R = 1.5/N$ (150% DB method).

Tabulate the annual depreciation amount and BV for each year.

Solution

Annual depreciation, cumulative depreciation, and BV are determined by using Equations (7-6), (7-7), and (7-8), respectively. Sample calculations for year six are as follows:

(a)

$$R = 2/10 = 0.2,$$

$$d_6 = \$4{,}000(1-0.2)^5(0.2) = \$262.14,$$

$$d_6^* = \$4{,}000[1-(1-0.2)^6] = \$2{,}951.42,$$

$$BV_6 = \$4{,}000(1-0.2)^6 = \$1{,}048.58.$$

(b)

$$R = 1.5/10 = 0.15,$$

$$d_6 = \$4{,}000(1-0.15)^5(0.15) = \$266.22,$$

$$d_6^* = \$4{,}000[1-(1-0.15)^6] = \$2{,}491.40,$$

$$BV_6 = \$4{,}000(1-0.15)^6 = \$1{,}508.60.$$

The depreciation and BV amounts for each year, when $R = 2/N = 0.2$, are shown in the following table:

200% DB Method Only		
EOY, k	d_k	BV_k
0	—	$4,000
1	$800	3,200
2	640	2,560
3	512	2,048
4	409.60	1,638.40
5	327.68	1,310.72
6	262.14	1,048.58
7	209.72	838.86
8	167.77	671.09
9	134.22	536.87
10	107.37	429.50

7.3.3 DB with Switchover to SL

Because the DB method never reaches a BV of zero, it is permissible to switch from this method to the SL method so that an asset's BV_N will be zero (or some other determined amount, such as SV_N). Also, this method is used in calculating the MACRS recovery rates (shown later in Table 7-3).

Table 7-1 illustrates a switchover from double DB depreciation to SL depreciation for Example 7-2. The switchover occurs in the year in which a larger depreciation amount is obtained from the SL method. From Table 7-1, it is apparent that $d_6 = \$262.14$. The BV at the end of year six (BV_6) is $1,048.58. Additionally, observe that BV_{10} is $4,000 − $3,570.50 = $429.50 without switchover to the SL method in Table 7-1. With switchover, BV_{10} equals zero. It is clear that this asset's d_k, d_k^*, and BV_k in years 7 through 10 are established from the SL method, which permits the full cost basis to be depreciated over the 10-year recovery period.

7.3.4 Units-of-Production Method

All the depreciation methods discussed to this point are based on elapsed time (years) on the theory that the decrease in value of property is mainly a function of time. When the decrease in value is mostly a function of use, depreciation may be based on a method not expressed in terms of years. The units-of-production method is normally used in this case.

The 200% DB Method with Switchover to the SL Method (Example 7-2)

	Depreciation Method			
	(1)	(2)	(3)	(4)
Year, k	Beginning-of-Year BV[a]	200% DB Method[b]	SL Method[c]	Depreciation Amount Selected[d]
1	$4,000.00	$800.00	>$400.00	$800.00
2	3,200.00	640.00	>355.56	640.00
3	2,560.00	512.00	>320.00	512.00
4	2,048.00	409.60	>292.57	409.60
5	1,638.40	327.68	>273.07	327.68
6	1,310.72	262.14	=262.14	262.14 (switch)
7	1,048.58	209.72	<262.14	262.14
8	786.44	167.77	<262.14	262.14
9	524.30	134.22	<262.14	262.14
10	262.16	107.37	<262.14	262.14
		$3,570.50		$4,000.00

[a] Column 1 for year k less column 4 for year k equals the entry in column 1 for year $k+1$.
[b] 200%(= 2/10) of column 1.
[c] Column 1 minus estimated SV_N divided by the remaining years from the beginning of the year through the 10th year.
[d] Select the larger amount in column 2 or column 3.

This method results in the cost basis (minus final SV) being allocated equally over the estimated number of units produced during the useful life of the asset. The depreciation rate is calculated as

$$\text{Depreciation per unit of production} = \frac{B - SV_N}{(\text{Estimated lifetime production units})}. \quad (7\text{-}9)$$

EXAMPLE 7-3 **Depreciation Based on Activity**

A piece of equipment used in a business has a basis of $50,000 and is expected to have a $10,000 SV when replaced after 30,000 hours of use. Find its depreciation rate per hour of use, and find its BV after 10,000 hours of operation.

Solution

$$\text{Depreciation per unit of production} = \frac{\$50{,}000 - \$10{,}000}{30{,}000 \text{ hours}} = \$1.33 \text{ per hour.}$$

$$\text{After 10,000 hours, BV} = \$50{,}000 - \frac{\$1.33}{\text{hour}}(10{,}000 \text{ hours}), \text{ or BV} = \$36{,}700.$$

7.4 The Modified Accelerated Cost Recovery System

As we discussed in Section 7.2.1, the MACRS was created by TRA 86 and is now the principal method for computing depreciation deductions for property in engineering projects. MACRS applies to most tangible depreciable property placed in service after December 31, 1986. Examples of assets that cannot be depreciated under MACRS are property you elect to exclude because it is to be depreciated under a method that is not based on a term of years (units-of-production method) and intangible property. Previous depreciation methods have required estimates of useful life (N) and SV at the end of useful life (SV_N). Under MACRS, however, SV_N is defined to be zero, and useful life estimates are not used directly in calculating depreciation amounts.

MACRS consists of two systems for computing depreciation deductions. The main system is called the *General Depreciation System (GDS)*, and the second system is called the *Alternative Depreciation System (ADS)*.

In general, ADS provides a longer recovery period and uses only the SL method of depreciation. Property that is placed in any tax-exempt use and property used predominantly outside the United States are examples of assets that must be depreciated under ADS. Any property that qualifies under GDS, however, can be depreciated under ADS, if elected.

When an asset is depreciated under MACRS, the following information is needed before depreciation deductions can be calculated:

1. The cost basis (B)
2. The date the property was placed in service
3. The property class and recovery period
4. The MACRS depreciation method to be used (GDS or ADS)
5. The time convention that applies (half year)

The first two items were discussed in Section 7.2. Items 3 through 5 are discussed in the sections that follow.

7.4.1 Property Class and Recovery Period

Under MACRS, tangible depreciable property is categorized (organized) into asset classes. The property in each asset class is then assigned *a class life, GDS recovery period (and property class), and ADS recovery period.* For our use, a partial listing of depreciable assets used in business is provided in Table 7-2. The types of depreciable property grouped together are identified in the second column. Then the class life, GDS recovery period (and property class), and ADS recovery period (all in years) for these assets are listed in the remaining three columns.

Under the GDS, the basic information about property classes and recovery periods is as follows:

1. Most tangible personal property is assigned to one of six *personal property classes* (3-, 5-, 7-, 10-, 15-, and 20-year property). The personal property class (in years) is the same as the *GDS recovery period.* Any depreciable personal property that

MACRS Class Lives and Recovery Periods[a]

Asset Class	Description of Assets	Class Life	Recovery Period GDS[b]	Recovery Period ADS
00.11	Office furniture and equipment	10	7	10
00.12	Information systems, including computers	6	5	5
00.22	Automobiles, taxis	3	5	5
00.23	Buses	9	5	9
00.241	Light general purpose trucks	4	5	5
00.242	Heavy general purpose trucks	6	5	6
00.26	Tractor units for use over the road	4	3	4
01.1	Agriculture	10	7	10
10.0	Mining	10	7	10
13.2	Production of petroleum and natural gas	14	7	14
13.3	Petroleum refining	16	10	16
15.0	Construction	6	5	6
22.3	Manufacture of carpets	9	5	9
24.4	Manufacture of wood products and furniture	10	7	10
28.0	Manufacture of chemicals and allied products	9.5	5	9.5
30.1	Manufacture of rubber products	14	7	14
32.2	Manufacture of cement	20	15	20
34.0	Manufacture of fabricated metal products	12	7	12
36.0	Manufacture of electronic components, products, and systems	6	5	6
37.11	Manufacture of motor vehicles	12	7	12
37.2	Manufacture of aerospace products	10	7	10
48.12	Telephone central office equipment	18	10	18
49.13	Electric utility steam production plant	28	20	28
49.21	Gas utility distribution facilities	35	20	35
79.0	Recreation	10	7	10

[a] Partial listing abstracted from *How to Depreciate Property*, IRS Publication 946, Tables B-1 and B-2, 2006.
[b] Also the *GDS property class*.

does not "fit" into one of the defined asset classes is depreciated as being in the seven-year property class.

2. Real property is assigned to two *real property classes*: nonresidential real property and residential rental property.
3. The *GDS recovery period* is 39 years for nonresidential real property (31.5 years if placed in service before May 13, 1993) and 27.5 years for residential real property.

The following is a summary of basic information for the ADS:

1. For tangible personal property, the *ADS recovery period* is shown in the last column on the right of Table 7-2 (and is normally the same as the class life of the property; there are exceptions such as those shown in asset classes 00.12 and 00.22).

2. Any tangible personal property that does not fit into one of the asset classes is depreciated under a 12-year ADS recovery period.
3. The *ADS recovery period* for nonresidential real property is 40 years.

The use of these rules under the MACRS is discussed further in the next section.

7.4.2 Depreciation Methods, Time Convention, and Recovery Rates

The primary methods used under MACRS for calculating the depreciation deductions over the recovery period of an asset are summarized as follows:

1. GDS 3-, 5-, 7-, and 10-year personal property classes: The 200% DB method, which switches to the SL method when that method provides a greater deduction. The DB method with switchover to SL was illustrated in Section 7.3.3.
2. GDS 15- and 20-year personal property classes: The 150% DB method, which switches to the SL method when that method provides a greater deduction.
3. GDS nonresidential real and residential rental property classes: The SL method over the fixed GDS recovery periods.
4. ADS: The SL method for both personal and real property over the fixed ADS recovery periods.

A *half-year time convention* is used in MACRS depreciation calculations for tangible personal property. This means that all assets placed in service during the year are treated as if use began in the middle of the year, and one-half year of depreciation is allowed. When an asset is disposed of, the half-year convention is also allowed. *If the asset is disposed of before the full recovery period is used, then only half of the normal depreciation deduction can be taken for that year.*

The GDS *recovery rates* (r_k) for the six personal property classes that we will use in our depreciation calculations are listed in Table 7-3. The GDS personal property rates in Table 7-3 include the half-year convention, as well as switchover from the DB method to the SL method when that method provides a greater deduction. Note that, if an asset is disposed of in year $N+1$, the final BV of the asset will be zero. Furthermore, there are $N+1$ recovery rates shown for each GDS property class for a property class of N years.

The depreciation deduction (d_k) for an asset under MACRS (GDS) is computed with

$$d_k = r_k \cdot B; \quad 1 \leq k \leq N+1, \qquad (7\text{-}10)$$

where r_k = recovery rate for year k from Table 7-3.

The information in Table 7-4 provides a summary of the principal features of GDS under MACRS. Included are some selected special rules about depreciable assets. A flow diagram for computing depreciation deductions under MACRS is shown in Figure 7-1. As indicated in the figure, an important choice is whether the main GDS is to be used for an asset or whether ADS is elected instead (or required). Normally, the choice would be to use the GDS for calculating the depreciation deductions.

GDS Recovery Rates (r_k) for the Six Personal Property Classes

Year	3-year[a]	5-year[a]	7-year[a]	10-year[a]	15-year[b]	20-year[b]
	Recovery Period (and Property Class)					
1	0.3333	0.2000	0.1429	0.1000	0.0500	0.0375
2	0.4445	0.3200	0.2449	0.1800	0.0950	0.0722
3	0.1481	0.1920	0.1749	0.1440	0.0855	0.0668
4	0.0741	0.1152	0.1249	0.1152	0.0770	0.0618
5		0.1152	0.0893	0.0922	0.0693	0.0571
6		0.0576	0.0892	0.0737	0.0623	0.0528
7			0.0893	0.0655	0.0590	0.0489
8			0.0446	0.0655	0.0590	0.0452
9				0.0656	0.0591	0.0447
10				0.0655	0.0590	0.0447
11				0.0328	0.0591	0.0446
12					0.0590	0.0446
13					0.0591	0.0446
14					0.0590	0.0446
15					0.0591	0.0446
16					0.0295	0.0446
17						0.0446
18						0.0446
19						0.0446
20						0.0446
21						0.0223

Source: IRS Publication 946. *Depreciation*. Washington, D.C.: U.S. Government Printing Office, for 2006 tax returns.

[a] These rates are determined by applying the 200% DB method (with switchover to the SL method) to the recovery period with the half-year convention applied to the first and last years. Rates for each period must sum to 1.0000.

[b] These rates are determined with the 150% DB method instead of the 200% DB method (with switchover to the SL method) and are rounded off to four decimal places.

EXAMPLE 7-4

MACRS Depreciation with GDS

A firm purchased and placed in service a new piece of semiconductor manufacturing equipment. The cost basis for the equipment is $100,000. Determine

(a) the depreciation charge permissible in the fourth year,

(b) the BV at the end of the fourth year,

(c) the cumulative depreciation through the third year,

(d) the BV at the end of the fifth year if the equipment is disposed of at that time.

MACRS (GDS) Property Classes and Primary Methods for Calculating Depreciation Deductions

GDS Property Class and Depreciation Method	Class Life (Useful Life)	Special Rules
3-year, 200% DB with switchover to SL	Four years or less	Includes some race horses and tractor units for over-the-road use.
5-year, 200% DB with switchover to SL	More than 4 years to less than 10	Includes cars and light trucks, semiconductor manufacturing equipment, qualified technological equipment, computer-based central office switching equipment, some renewable and biomass power facilities, and research and development property.
7-year, 200% DB with switchover to SL	10 years to less than 16	Includes single-purpose agricultural and horticultural structures and railroad track. Includes office furniture and fixtures, and property not assigned to a property class.
10-year, 200% DB with switchover to SL	16 years to less than 20	Includes vessels, barges, tugs, and similar water transportation equipment.
15-year, 150% DB with switchover to SL	20 years to less than 25	Includes sewage treatment plants, telephone distribution plants, and equipment for two-way voice and data communication.
20-year, 150% DB with switchover to SL	25 years or more	Excludes real property of 27.5 years or more. Includes municipal sewers.
27.5 year, SL	N/A	Residential rental property.
39-year, SL	N/A	Nonresidential real property.

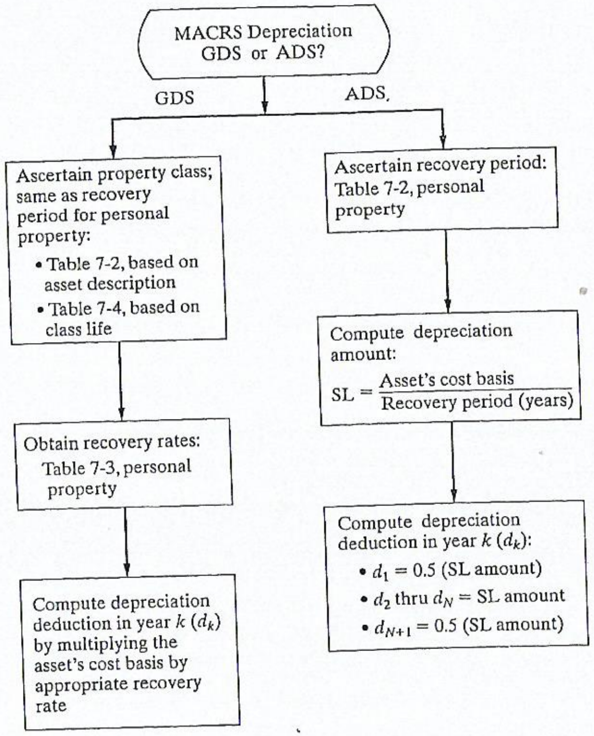
Source: IRS Publication 946. *Depreciation*. Washington, D.C.: U.S. Government Printing Office, for 2006 tax returns.

Solution

From Table 7-2, it may be seen that the semiconductor (electronic) manufacturing equipment has a class life of six years and a GDS recovery period of five years. The recovery rates that apply are given in Table 7-3.

(a) The depreciation deduction, or cost-recovery allowance, that is allowable in year four (d_4) is 0.1152 ($100,000) = $11,520.

Figure 7-1
Flow Diagram for Computing Depreciation Deductions under MACRS



(b) The BV at the end of year four (BV_4) is the cost basis less depreciation charges in years one through four:

$$BV_4 = \$100{,}000 - \$100{,}000(0.20 + 0.32 + 0.192 + 0.1152)$$
$$= \$17{,}280.$$

(c) Accumulated depreciation through year three, d_3^*, is the sum of depreciation amounts in years one through three:

$$d_3^* = d_1 + d_2 + d_3$$
$$= \$100{,}000(0.20 + 0.32 + 0.192)$$
$$= \$71{,}200.$$

(d) The depreciation deduction in year five can only be (0.5)(0.1152)(\$100,000) = \$5,760 when the equipment is disposed of prior to year six. Thus, the BV at the end of year five is $BV_4 - \$5{,}760 = \$11{,}520$.

EXAMPLE 7-5 MACRS with a Trade-in

In May 2011, your company traded in a computer and peripheral equipment, used in its business, that had a BV at that time of \$25,000. A new, faster computer system having a fair MV of \$400,000 was acquired. Because the vendor accepted the older computer as a trade-in, a deal was agreed to whereby your company would pay \$325,000 cash for the new computer system.

(a) What is the GDS property class of the new computer system?

(b) How much depreciation can be deducted each year based on this class life? (Refer to Figure 7-1.)

Solution

(a) The new computer is in Asset Class 00.12 and has a class life of six years. (See Table 7-2.) Hence, its GDS property class and recovery period are five years.

(b) The cost basis for this property is \$350,000, which is the sum of the \$325,000 cash price of the computer and the \$25,000 BV remaining on the trade-in. (In this case, the trade-in was treated as a nontaxable transaction.)

MACRS (GDS) rates that apply to the \$350,000 cost basis are found in Table 7-3. An allowance (half-year) is built into the year-one rate, so it does not matter that the computer was purchased in May 2011 instead of, say, November 2011. The depreciation deductions (d_k) for 2011 through 2016 are computed using Equation (7-10) and Table 7-3.

Property	Date Placed in Service	Cost Basis	Class Life	MACRS (GDS) Recovery Period
Computer System	May 2005	\$350,000	6 years	5 years

Year	Depreciation Deductions
2011	0.20 × \$350,000 = \$70,000
2012	0.32 × 350,000 = 112,000
2013	0.192 × 350,000 = 67,200
2014	0.1152 × 350,000 = 40,320
2015	0.1152 × 350,000 = 40,320
2016	0.0576 × 350,000 = 20,160
	Total \$350,000

From Example 7-5, we can conclude that Equation 7-11 is true from the buyer's viewpoint when property of the same type and class is exchanged:

$$\text{Basis} = \text{Actual cash cost} + \text{BV of the trade-in.} \tag{7-11}$$

To illustrate Equation (7-11), suppose your company has operated an optical character recognition (OCR) scanner for two years. Its BV now is $35,000, and its fair MV is $45,000. The company is considering a new OCR scanner that costs $105,000. Ordinarily, you would trade the old scanner for the new one and pay the dealer $60,000. The basis ($B$) for depreciation is then $60,000 + $35,000 = $95,000.*

EXAMPLE 7-6

MACRS Depreciation with ADS

A large manufacturer of sheet metal products in the Midwest purchased and placed in service a new, modern, computer-controlled flexible manufacturing system for $3.0 million. Because this company would not be profitable until the new technology had been in place for several years, it elected to utilize the ADS under MACRS in computing its depreciation deductions. Thus, the company could slow down its depreciation allowances in hopes of postponing its income tax advantages until it became a profitable concern. What depreciation deductions can be claimed for the new system?

Solution

From Table 7-2, the ADS recovery period for a manufacturer of fabricated metal products is 12 years. Under ADS, the SL method with no SV is applied to the 12-year recovery period, using the half-year convention. Consequently, the depreciation in year one would be

$$\frac{1}{2}\left(\frac{\$3{,}000{,}000}{12}\right) = \$125{,}000.$$

Depreciation deductions in years 2 through 12 would be $250,000 each year, and depreciation in year 13 would be $125,000. Notice how the half-year convention extends depreciation deductions over 13 years ($N + 1$).

7.5 A Comprehensive Depreciation Example

We now consider an asset for which depreciation is computed by the historical and MACRS (GDS) methods previously discussed. Be careful to observe the differences in the mechanics of each method, as well as the differences in the annual depreciation amounts themselves. Also, we compare the present worths (PWs) at $k = 0$ of selected depreciation methods when MARR = 10% per year. As we shall see later in this chapter, depreciation methods that result in larger PWs (of the depreciation amounts) are preferred by a firm that wants to reduce the present worth of its *income taxes paid* to the government.

* The *exchange price* of the OCR scanner is $105,000 − $45,000(= actual cash cost). Equation (7-11) prevents exaggerated cost bases to be claimed for new assets having large "sticker prices" compared with their exchange price.

EXAMPLE 7-7

Comparison of Depreciation Methods

The La Salle Bus Company has decided to purchase a new bus for $85,000 with a trade-in of their old bus. The old bus has a BV of $10,000 at the time of the trade-in. The new bus will be kept for 10 years before being sold. Its estimated SV at that time is expected to be $5,000.

First, we must calculate the cost basis. The basis is the original purchase price of the bus plus the BV of the old bus that was traded in [Equation (7-11)]. Thus, the basis is $85,000 + $10,000, or $95,000. We need to look at Table 7-2 and find buses, which are asset class 00.23. Hence, we find that buses have a nine-year class (useful) life, over which we depreciate the bus with historical methods discussed in Section 7.3, and a five-year GDS recovery period.

Solution: SL Method

For the SL method, we use the class life of 9 years, even though the bus will be kept for 10 years. By using Equations (7-2) and (7-4), we obtain the following information:

$$d_k = \frac{\$95{,}000 - \$5{,}000}{9 \text{ years}} = \$10{,}000, \quad \text{for } k = 1 \text{ to } 9.$$

	SL Method	
EOY k	d_k	BV_k
0	—	$95,000
1	$10,000	85,000
2	10,000	75,000
3	10,000	65,000
4	10,000	55,000
5	10,000	45,000
6	10,000	35,000
7	10,000	25,000
8	10,000	15,000
9	10,000	5,000

Notice that no depreciation was taken after year nine because the class life was only nine years. Also, the final BV was the estimated SV, and the BV will remain at $5,000 until the bus is sold.

Solution: DB Method

To demonstrate this method, we will use the 200% DB equations. With Equations (7-6) and (7-8), we calculate the following:

$$R = 2/9 = 0.2222;$$

$$d_1 = \$95{,}000(0.2222) = \$21{,}111;$$

$$d_5 = \$95{,}000(1 - 0.2222)^4(0.2222) = \$7{,}726;$$
$$BV_5 = \$95{,}000(1 - 0.2222)^5 = \$27{,}040.$$

200% DB Method		
EOY k	d_k	BV_k
0	—	\$95,000
1	\$21,111	73,889
2	16,420	57,469
3	12,771	44,698
4	9,932	34,765
5	7,726	27,040
6	6,009	21,031
7	4,674	16,357
8	3,635	12,722
9	2,827	9,895

Solution: DB with Switchover to SL Depreciation

To illustrate the mechanics of Table 7-1 for this example, we first specify that the bus will be depreciated by the 200% DB method ($R = 2/N$). Because DB methods never reach a zero BV, suppose that we further specify that a switchover to SL depreciation will be made to ensure a BV of \$5,000 at the end of the vehicle's nine-year class life.

EOY k	Beginning-of-Year BV	200% DB Method	SL Method ($BV_9 = \$5{,}000$)	Depreciation Amount Selected
1	\$95,000	\$21,111	\$10,000	\$21,111
2	73,889	16,420	8,611	16,420
3	57,469	12,771	7,496	12,771
4	44,698	9,933	6,616	9,933
5	34,765	7,726	5,953	7,726
6	27,040	6,009	5,510	6,009
7	21,031	4,674	5,344	5,344[a]
8	15,687	3,635	5,344	5,344
9	10,344	2,827	5,344	5,344

[a] Switchover occurs in year seven.

Solution: MACRS (GDS) with Half-Year Convention

To demonstrate the GDS with the half-year convention, we will change the La Salle bus problem so that the bus is now sold in year five for Part (a) and in year six for Part (b).

(a) Selling bus in year five:

EOY k	Factor	d_k	BV_k
0	—	—	\$95,000
1	0.2000	\$19,000	76,000
2	0.3200	30,400	45,600
3	0.1920	18,240	27,360
4	0.1152	10,944	16,416
5	0.0576	5,472	10,944

(b) Selling bus in year six:

EOY k	Factor	d_k	BV_k
0	—	—	\$95,000
1	0.2000	\$19,000	76,000
2	0.3200	30,400	45,600
3	0.1920	18,240	27,360
4	0.1152	10,944	16,416
5	0.1152	10,944	5,472
6	0.0576	5,472	0

Notice that, when we sold the bus in year five before the recovery period had ended, we took only half of the normal depreciation. The other years (years one through four) were not changed. When the bus was sold in year six, at the end of the recovery period, we did not divide the last year amount by two.

Selected methods of depreciation, illustrated in Example 7-7, are compared in Figure 7-2. In addition, the PW (10%) of each method is shown in Figure 7-2. Because large PWs of depreciation deductions are generally viewed as desirable, it is clear that the MACRS method is very attractive to profitable companies.

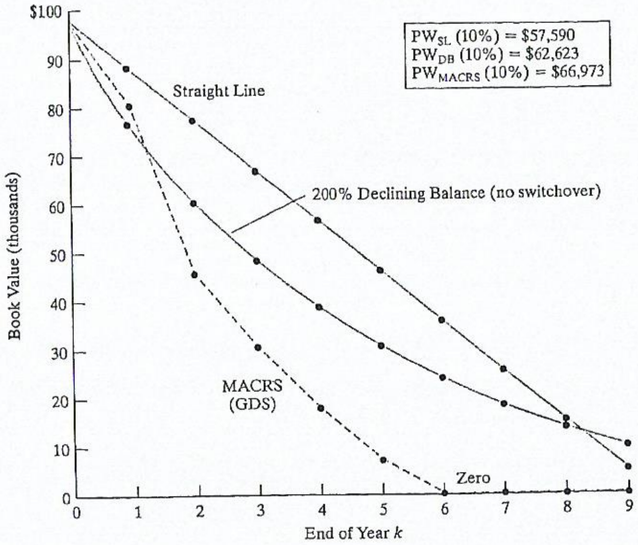


Figure 7-2 BV Comparisons for Selected Methods of Depreciation in Example 7-7 (*Note:* The bus is assumed to be sold in year six for the MACRS-GDS method.)

7.6 Introduction to Income Taxes

Up to this point, there has been no consideration of income taxes in our discussion of engineering economy, except for the influence of depreciation and other types of deductions. By not complicating our studies with income tax effects, we have placed primary emphasis on basic engineering economy principles and methodology. There, however, is a wide variety of capital investment problems in which income taxes do affect the choice among alternatives, and after-tax studies are essential.*

In the remainder of this chapter, we shall be concerned with how income taxes affect a project's estimated cash flows. Income taxes resulting from the profitable operation of a firm are usually taken into account in evaluating engineering projects. The reason is quite simple: Income taxes associated with a proposed project may represent a major cash outflow that should be considered together with other cash inflows and outflows in assessing the overall economic profitability of that project.

There are other taxes discussed in Section 7.6.1 that are not directly associated with the income-producing capability of a new project, but they are usually

* Some U.S. firms do not pay income taxes, but this is not a valid reason for ignoring income taxes. We assume that most firms have a positive taxable income in their overall operations.

negligible when compared with federal and state income taxes. *When other types of taxes are included in engineering economy studies, they are normally deducted from revenue, as any other operating expense would be, in determining the before-tax cash flow (BTCF) that we considered in Chapters 5 and 6.*

The mystery behind the sometimes complex computation of income taxes is reduced when we recognize that income taxes paid are just another type of expense, and income taxes saved (through depreciation, expenses, and direct tax credits) are identical to other kinds of reduced expenses (e.g., savings in maintenance expenses).

The basic concepts underlying federal and state income tax laws and regulations that apply to most economic analyses of capital investments generally can be understood and applied without difficulty. This discussion of income taxes is not intended to be a comprehensive treatment of the subject. Rather, we utilize some of the more important provisions of the federal Tax Reform Act of 1986 (TRA 86), followed by illustrations of a general procedure for computing the net ATCF for an engineering project and conducting after-tax economic analyses. Where appropriate, important changes to TRA 86 enacted by the Omnibus Budget Reconciliation Act of 1993 (OBRA 93) and the Taxpayer Relief Act of 1997 are also included in this chapter.

7.6.1 Distinctions between Different Types of Taxes

Before discussing the consequences of income taxes in engineering economy studies, we need to distinguish between income taxes and several other types of taxes:

1. ***Income taxes*** are assessed as a function of gross revenues minus allowable deductions. They are levied by the federal, most state, and occasionally municipal governments.
2. ***Property taxes*** are assessed as a function of the value of property owned, such as land, buildings, equipment, and so on, and the applicable tax rates. Hence, they are independent of the income or profit of a firm. They are levied by municipal, county, or state governments.
3. ***Sales taxes*** are assessed on the basis of purchases of goods or services and are thus independent of gross income or profits. They are normally levied by state, municipal, or county governments. Sales taxes are relevant in engineering economy studies only to the extent that they add to the cost of items purchased.
4. ***Excise taxes*** are federal taxes assessed as a function of the sale of certain goods or services often considered nonnecessities and are hence independent of the income or profit of a business. Although they are usually charged to the manufacturer or original provider of the goods or services, a portion of the cost is passed on to the purchaser.

7.6.2 The Before-Tax and After-Tax Minimum Attractive Rates of Return

In the preceding chapters, we have treated income taxes as if they are not applicable, or we have taken them into account, in general, by using a before-tax MARR, which is larger than the after-tax MARR. An approximation of the before-tax MARR requirement, which includes the effect of income taxes, for studies involving only BTCF can be obtained from the following relationship:

$$(\text{Before-tax MARR})[(1 - \text{Effective income tax rate})] \cong \text{After-tax MARR}.$$

Thus,

$$\text{Before-tax MARR} \approx \frac{\text{After-tax MARR}}{(1 - \text{Effective income tax rate})}. \quad (7\text{-}12)$$

Determining the effective income tax rate for a firm is discussed in Section 7.7.

This approximation is exact if the asset is nondepreciable and there are no gains or losses on disposal, tax credits, or other types of deductions involved. Otherwise, these factors affect the amount and timing of income tax payments, and some degree of error is introduced into the relationship in Equation (7-12).

7.6.3 The Interest Rate to Use in After-Tax Studies

Any practical definition of a MARR should serve as a guide for investment policy to attain the goals toward which a firm is, or will be, striving. One obvious goal is to meet shareholder expectations. Clearly, long-term goals and capital structure will be changing as a firm matures. In this regard, it is widely accepted that in after-tax engineering economy studies, the MARR should be *at least* the tax-adjusted weighted average cost of capital (WACC):

$$\text{WACC} = \lambda(1 - t)i_b + (1 - \lambda)e_a. \quad (7\text{-}13)$$

Here, λ = fraction of a firm's pool of capital that is borrowed from lenders,
t = effective income tax rate as a decimal,
i_b = before-tax interest paid on borrowed capital,
e_a = after-tax cost of equity capital.

The adjustment on the borrowed capital component of WACC, $(1 - t)i_b$, accounts for the fact that interest on borrowed capital is tax deductible.

Often a value higher than the WACC is assigned to the after-tax MARR to reflect the opportunity cost of capital, perceived risk and uncertainty of projects being evaluated, and policy issues such as organizational growth and shareholder satisfaction. We are now in a position to understand how a firm's after-tax MARR is determined and why its value may change substantially over time. To summarize, suppose a firms' WACC is 10% per year. The after-tax MARR may be set at 15% per year to better reflect the business opportunities that are available for capital investment. If the firm's effective income tax rate is 40%, the approximate value of the before-tax MARR is 15%/(1 − 0.4) = 25% per year.

7.6.4 Taxable Income of Corporations (Business Firms)

At the end of each tax year, a corporation must calculate its net (i.e., taxable) before-tax income or loss. Several steps are involved in this process, beginning with the calculation of *gross income*. The corporation may deduct from gross income all ordinary and necessary operating expenses to conduct the business *except capital investments*. Deductions for depreciation are permitted each tax

period as a means of consistently and systematically recovering capital investment. Consequently, allowable expenses and deductions may be used to determine taxable income:

$$\text{Taxable income} = \text{Gross income} - \text{All expenses except capital investments} - \text{Depreciation deductions}. \quad (7\text{-}14)$$

EXAMPLE 7-9 **Determination of Taxable Income**

A company generates $1,500,000 of gross income during its tax year and incurs operating expenses of $800,000. Property taxes on business assets amount to $48,000. The total depreciation deductions for the tax year equal $114,000. What is the taxable income of this firm?

Solution

Based on Equation (7-14), this company's taxable income for the tax year would be

$$\$1{,}500{,}000 - \$800{,}000 - \$48{,}000 - \$114{,}000 = \$538{,}000.$$

7.7 The Effective (Marginal) Corporate Income Tax Rate

The federal corporate income tax rate structure in 2006 is shown in Table 7-5. Depending on the taxable income bracket that a firm is in for a tax year, the marginal federal rate can vary from 15% to a maximum of 39%. Note, however, that the weighted average tax rate at taxable income = $335,000 is 34%, and the weighted average tax rate at taxable income = $18,333,333 is 35%. Therefore, if a corporation has a taxable income for a tax year *greater than* $18,333,333, federal taxes are computed by using a flat rate of 35%.

Corporate Federal Income Tax Rates (2006)

If Taxable Income Is:		The Tax Is:	
Over	But Not Over		of the Amount Over
0	$50,000	15%	0
$50,000	75,000	$7,500 + 25%	$50,000
75,000	100,000	13,750 + 34%	75,000
100,000	335,000	22,250 + 39%	100,000
335,000	10,000,000	113,900 + 34%	335,000
10,000,000	15,000,000	3,400,000 + 35%	10,000,000
15,000,000	18,333,333	5,150,000 + 38%	15,000,000
18,333,333	....	35%	0

Source: Tax Information on Corporations, IRS Publication 542, 2006.

EXAMPLE 7-9

Calculating Income Taxes

Suppose that a firm for a tax year has a gross income of $5,270,000, expenses (excluding capital) of $2,927,500, and depreciation deductions of $1,874,300. What would be its taxable income and federal income tax for the tax year, based on Equation (7-14) and Table 7-5?

Solution

Taxable income = Gross income − Expenses − Depreciation deductions

= $5,270,000 − $2,927,500 − $1,874,300

= $468,200

Income tax =	15% of first $50,000	$7,500
	+ 25% of the next $25,000	6,250
	+ 34% of the next $25,000	8,500
	+ 39% of the next $235,000	91,650
	+ 34% of the next $133,200	45,288
	Total	$159,188

The total tax liability in this case is $159,188. As an added note, we could have used a flat rate of 34% in this example because the federal weighted average tax rate at taxable income = $335,000 is 34%. The remaining $133,200 of taxable income above this amount is in a 34% tax bracket (Table 7-5). So we have 0.34($468,200) = $159,188.

Although the tax laws and regulations of most of the states (and some municipalities) with income taxes have the same basic features as the federal laws and regulations, there is significant variation in income tax rates. State income taxes are in most cases much less than federal taxes and often can be closely approximated as ranging from 6% to 12% of taxable income. No attempt will be made here to discuss the details of state income taxes. To illustrate the calculation of an effective income tax rate (t) for a large corporation based on the consideration of both federal and state income taxes, however, assume that the applicable federal income tax rate is 35% and the state income tax rate is 8%. Further assume the common case in which taxable income is computed the same way for both types of taxes, except that state income taxes are deductible from taxable income for federal tax purposes, but federal income taxes are not deductible from taxable income for state tax purposes. Based on these assumptions, the general expression for the effective income tax rate is

$$t = \text{State rate} + \text{Federal rate}(1 - \text{State rate}), \text{ or} \quad (7\text{-}15)$$

$$t = \text{Federal rate} + (1 - \text{Federal rate})(\text{State rate}). \quad (7\text{-}16)$$

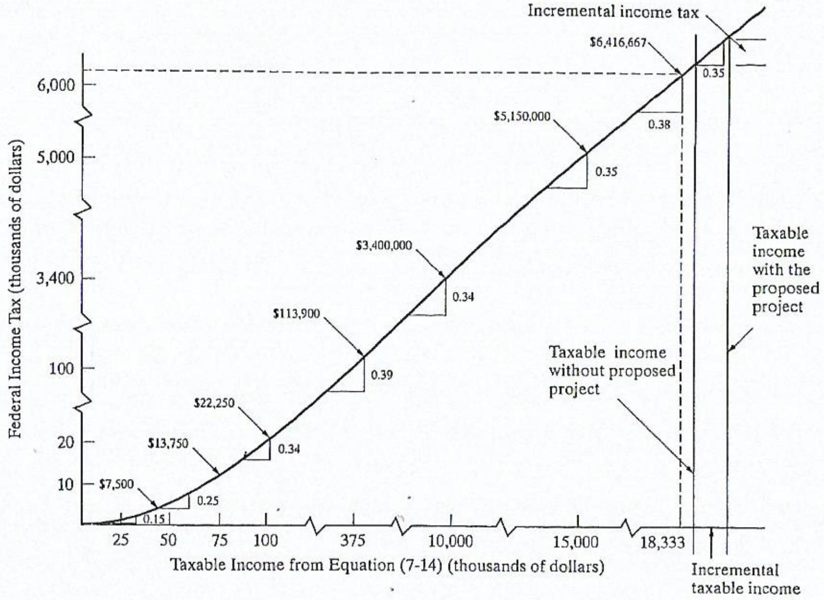


Figure 7-3 The Federal Income Tax Rates for Corporations (Table 7-5) with Incremental Income Tax for a Proposed Project (assumes, in this case, corporate taxable income without project > $18,333,333)

In this example, the effective income tax rate for the corporation would be

$$t = 0.08 + 0.35(1 - 0.08) = 0.402, \text{ or approximately } 40\%.$$

In this chapter, we will often use an effective corporate income tax rate of *approximately* 40% as a representative value that includes state income taxes.

The effective income tax rate on *increments* of taxable income is of importance in engineering economy studies. This concept is illustrated in Figure 7-3, which plots the federal income tax rates and brackets listed in Table 7-5 and shows the added (incremental) taxable income and federal income taxes that would result from a proposed engineering project (shaded in Figure 7-3). In this case, the corporation is assumed to have a taxable income greater than $18,333,333 for their tax year. The same concept, however, applies to a smaller firm with less taxable income for its tax year, which is illustrated in Example 7-10.

EXAMPLE 7-10

Project-Based Income Taxes

A small corporation is expecting an annual taxable income of $45,000 for its tax year. It is considering an additional capital investment of $100,000 in an engineering project, which is expected to create an added annual net cash flow (revenues minus expenses) of $35,000 and an added annual depreciation deduction of $20,000. What is the corporation's federal income tax liability

(a) without the added capital investment,

(b) with the added capital investment?

Solution

(a)

Income Taxes	*Rate*	*Amount*
On first $45,000	15%	$6,750
	Total	$6,750

(b)

Taxable Income		
Before added investment		$45,000
+ added net cash flow		+35,000
− depreciation deduction		−20,000
	Taxable Income	$60,000

Income Taxes on $60,000	*Rate*	*Amount*
On first $50,000	15%	$7,500
On next $10,000	25%	2,500
	Total	$10,000

The increased income tax liability from the investment is $10,000 − $6,750 = $3,250.

As an added note, the change in tax liability can usually be determined more readily by an incremental approach. For instance, this example involved changing the taxable income from $45,000 to $60,000 as a result of the new investment. Thus, the change in income taxes for the tax year could be calculated as follows:

First $50,000 − $45,000 = $5,000 at 15% =	$750
Next $60,000 − $50,000 = $10,000 at 25% =	2,500
Total	$3,250

The average federal income tax rate on the additional $35,000 − $20,000 = $15,000 of taxable income is calculated as ($3,250/$15,000) = 0.2167, or 21.67%.

7.8 Gain (Loss) on the Disposal of an Asset

When a *depreciable asset* (tangible personal or real property, Section 7.2) is sold, the MV is seldom equal to its BV [Equation (7-1)]. In general, the gain (loss) on sale of depreciable property is the fair market value minus its book value at that time. That is,

$$[\text{Gain (loss) on disposal}]_N = MV_N - BV_N. \qquad (7\text{-}17)$$

When the sale results in a gain, it is often referred to as *depreciation recapture*. The tax rate for the gain (loss) on disposal of depreciable personal property is usually the same as for ordinary income or loss, which is the effective income tax rate, t.

When a *capital asset* is sold or exchanged, the gain (loss) is referred to as a *capital gain (loss)*. Examples of capital assets are stocks, bonds, gold, silver, and other metals, as well as real property such as a home. Because engineering economic analysis seldom involves an actual capital gain (loss), the more complex details of this situation are not discussed further.

EXAMPLE 7-11 **Tax Consequences of Selling an Asset**

A corporation sold a piece of equipment during the current tax year for $78,600. The accounting records show that its cost basis, B, is $190,000 and the accumulated depreciation is $139,200. Assume that the effective income tax rate as a decimal is 0.40 (40%). Based on this information, what is

(a) the gain (loss) on disposal,

(b) the tax liability (or credit) resulting from this sale,

(c) the tax liability (or credit) if the accumulated depreciation was $92,400 instead of $139,200?

Solution

(a) The BV at the time of sale is $190,000 − $139,200 = $50,800. Therefore, the gain on disposal is $78,600 − $50,800 = $27,800.

(b) The tax owed on this gain is −0.40($27,800) = −$11,120.

(c) With $d_k^* = \$92{,}400$, the BV at the time of sale is $190,000 − $92,400 = $97,600. The loss is $78,600 − $97,600 = −$19,000. The tax credit resulting from this loss on disposal is −0.40(−$19,000) = $7,600.

7.9 General Procedure for Making After-Tax Economic Analyses

After-tax economic analyses utilize the same profitability measures as do before-tax analyses. The only difference is that ATCFs are used in place of before-tax cash flows (BTCFs) by including expenses (or savings) to income taxes and then making equivalent worth calculations *using an after-tax MARR*. The tax rates and governing regulations may be complex and subject to changes, but, once those rates and regulations have been translated into their effect on ATCFs, the remainder of the after-tax analysis is relatively straightforward. To formalize the procedure, let

R_k = revenues (and savings) from the project (this is the cash inflow from the project during period k);

E_k = cash outflows during year k for deductible expenses;

d_k = sum of all noncash, or book, costs during year k, such as depreciation;
t = effective income tax rate on *ordinary* income (federal, state, and other), with t assumed to remain constant during the study period;
T_k = income tax consequences during year k;
ATCF_k = ATCF from the project during year k.

Because the taxable income is $(R_k - E_k - d_k)$, the *ordinary income tax consequences* during year k are computed with Equation (7-18):

$$T_k = -t(R_k - E_k - d_k). \tag{7-18}$$

Therefore, when $R_k > (E_k + d_k)$, a tax liability (i.e., negative cash flow) occurs. When $R_k < (E_k + d_k)$, a decrease in the tax amount (a credit) occurs.

In the economic analyses of engineering projects, ATCFs in year k can be computed in terms of BTCF_k (i.e., BTCF in year k):

$$\text{BTCF}_k = R_k - E_k. \tag{7-19}$$

Thus,*

$$\text{ATCF}_k = \text{BTCF}_k + T_k \tag{7-20}$$

$$= (R_k - E_k) - t(R_k - E_k - d_k)$$

$$= (1 - t)(R_k - E_k) + td_k. \tag{7-21}$$

Equation (7-21) shows that after income taxes, a revenue becomes $(1 - t)R_k$ and an expense becomes $(1 - t)E_k$. Note that the ATCF attributable to depreciation (a tax savings) is td_k in year k.

Tabular headings to facilitate the computation of ATCFs using Equations (7-18) and (7-21) are as follows:

Year	(A) BTCF	(B) Depreciation	(C) = (A) − (B) Taxable Income	(D) = −t(C) Cash Flow for Income Taxes	(E) = (A) + (D) ATCF
k	$R_k - E_k$	d_k	$R_k - E_k - d_k$	$-t(R_k - E_k - d_k)$	$(1 - t)(R_k - E_k) + td_k$

Column *A* consists of the same information used in before-tax analyses, namely the cash revenues (or savings) less the deductible expenses. Column *B* contains depreciation that can be claimed for tax purposes. Column C is the taxable income or the amount subject to income taxes. Column *D* contains the income taxes paid (or saved). Finally, column *E* shows the ATCFs to be used directly in after-tax economic analyses.

A note of caution concerning the definition of BTCFs (and ATCFs) for projects is in order at this point. BTCF is defined to be annual revenue (or savings) attributable to a project minus its annual cash expenses. These expenses should *not* include interest and other financial cash flows. The reason is that *project* cash flows should be analyzed separately from financial cash flows. Including interest expense with project cash flows is incorrect when the firm's pool of capital is being used to

* In Figure 7-4, we use $-t$ in column D, so algebraic subtraction of income taxes in Equation (7-20) is accomplished.

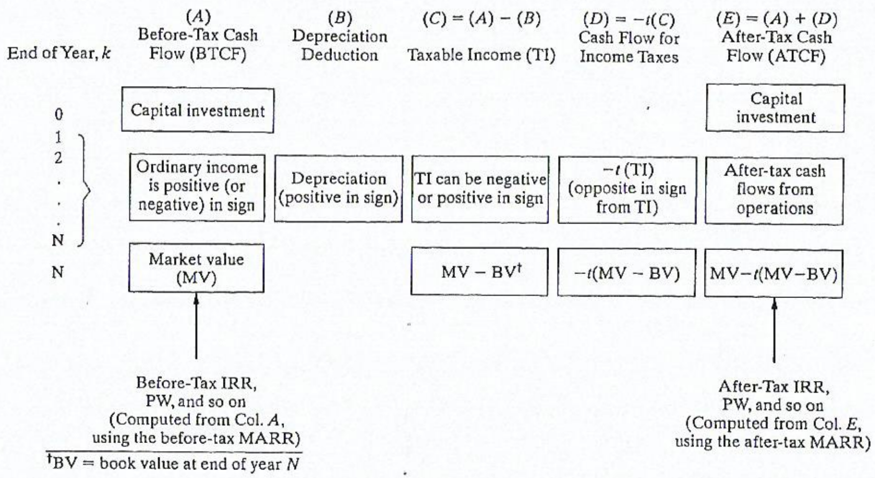


Figure 7-4 General Format (Worksheet) for After-Tax Analysis; Determining the ATCF

undertake an engineering project. Why? A firm's pool of capital consists of debt capital and equity capital. Because the MARR typically uses the weighted average cost of capital as its lower limit, discounting at the MARR for investments from the pool of capital takes account of the cost of debt capital (interest). Thus, there is no need to subtract interest expense in determining BTCFs—to do so would amount to double counting the interest expense associated with debt capital.

A summary of the process of determining ATCF during each year of an N-year study period is provided in Figure 7-4. The format of Figure 7-4 is used extensively throughout the remainder of this chapter, and it provides a convenient way to organize data in after-tax studies.

The column headings of Figure 7-4 indicate the arithmetic operations for computing columns C, D, and E when $k = 1, 2, \ldots, N$. When $k = 0$, capital investments are usually involved, and their tax treatment (if any) is illustrated in the examples that follow. The table should be used with the conventions of + for cash inflow or savings and − for cash outflow or opportunity forgone.

EXAMPLE 7-12 **PW of MACRS Depreciation Amounts**

Suppose that an asset with a cost basis of \$100,000 and an ADS recovery period of five years is being depreciated under the *Alternate Depreciation System (ADS) of MACRS*, as follows:

Year	1	2	3	4	5	6
Depreciation Deduction	\$10,000	\$20,000	\$20,000	\$20,000	\$20,000	\$10,000

If the firm's effective income tax rate remains constant at 40% during this six-year period, what is the PW of after-tax savings resulting from depreciation when MARR = 10% per year (after taxes)?

Solution

The PW of tax credits (savings) because of this depreciation schedule is

$$\text{PW}(10\%) = \sum_{k=1}^{6} 0.4d_k(1.10)^{-k} = \$4{,}000(0.9091) + \$8{,}000(0.8264)$$
$$+ \cdots + \$4{,}000(0.5645) = \$28{,}948.$$

EXAMPLE 7-13

After-Tax PW of an Asset

The asset in Example 7-12 is expected to produce net cash inflows (net revenues) of $30,000 per year during the six-year period, and its terminal MV is negligible. If the effective income tax rate is 40%, how much can a firm afford to spend for this asset and still earn the MARR? What is the meaning of any excess in affordable amount over the $100,000 cost basis given in Example 7-12? [Equals the after tax PW.]

Solution

After income taxes, the PW of net revenues is $(1-0.4)(\$30{,}000)(P/A, 10\%, 6) = \$18{,}000(4.3553) = \$78{,}395$. After adding to this the PW of tax savings computed in Example 7-12, the affordable amount is $107,343. Because the capital investment is $100,000, the net PW equals $7,343. This same result can be obtained by using the general format (worksheet) of Figure 7-4:

EOY	(A) BTCF	(B) Depreciation Deduction	(C) = (A) − (B) Taxable Income	(D) = − 0.4(C) Income Taxes	(E) = (A) + (D) ATCF
0	−$100,000				−$100,000
1	30,000	$10,000	$20,000	−$8,000	22,000
2	30,000	20,000	10,000	−4,000	26,000
3	30,000	20,000	10,000	−4,000	26,000
4	30,000	20,000	10,000	−4,000	26,000
5	30,000	20,000	10,000	−4,000	26,000
6	30,000	10,000	20,000	−8,000	22,000
					PW(10%) of ATCF = $7,343

EXAMPLE 7-14

PennCo Electric Utility Company

Here we refer back to the chapter opener where the question of the after-tax cost of electricity production was raised. For the PennCo plant, the annual fuel expense† will be $191,346,432 per year. The operating and maintenance expense will be $112,128,000 per year, and the carbon tax will be $73,805,052 per year. The total annual expense of operating the plant will therefore be $377,279,484. Armed with these cost estimates, we can calculate the after-tax cost of the plant by using the template of Figure 7-4 (all numbers are millions of dollars).

EOY	BTCF	Depreciation	Taxable Income	Cash Flow for Income Taxes	ATCF
0	−$1,120.000	—	—	—	−$1,120.000
1 − 30	−377.279	37.333*	−414.613	165.845	−211.434

*d_k = $1,120/30 = $37.333

The annual worth of the PennCo plant, assuming inflation is negligible, is $-\$1{,}120\ (A/P, 10\%, 30) - \$211.434$, which equals −$330.266 million. We can calculate the after-tax cost of a kilowatt-hour by dividing this number ($330,266,000) by the electrical output of 5,606,400,000 kWh. It is close to $0.06 per kWh.

† The details of the annual expense calculations are left as a student exercise in Problem 7-60.

7.10 Illustration of Computations of ATCFs

The following problems (Examples 7-15 through 7-19) illustrate the computation of ATCFs, as well as many common situations that affect income taxes. All problems include the assumption that income tax expenses (or savings) occur at the same time (year) as the revenue or expense that gives rise to the taxes. For purposes of comparing the effects of various situations, the after-tax IRR or PW is computed for each example. We can observe from the results of Examples 7-15, 7-16, and 7-17 that the faster (i.e., earlier) the depreciation deduction is, the more favorable the after-tax IRR and PW will become.

EXAMPLE 7-15

Computing After-Tax PW and IRR

Certain new machinery, when placed in service, is estimated to cost $180,000. It is expected to *reduce* net annual operating expenses by $36,000 per year for 10 years and to have a $30,000 MV at the end of the 10th year.

(a) Develop the ATCFs and BTCFs.

(b) Calculate the before-tax and after-tax IRR. Assume that the firm is in the federal taxable income bracket of $335,000 to $10,000,000 and that the state income tax rate is 6%. State income taxes are deductible from federal taxable income. This machinery is in the MACRS (GDS) five-year property class.

(c) Calculate the after-tax PW when the *after-tax* MARR = 10% per year.

In this example, the study period is 10 years, but the property class of the machinery is 5 years. Solve by hand and by spreadsheet.

Solution by Hand

(a) Table 7-6 applies the format illustrated in Figure 7-4 to calculate the BTCF and ATCF for this example. In column *D*, the effective income tax rate is very close to 0.38 [from Equation (7-15)] based on the information just provided.

(b) The before-tax IRR is computed from column *A*:

$$0 = -\$180{,}000 + \$36{,}000(P/A, i'\%, 10) + \$30{,}000(P/F, i'\%, 10).$$

By trial and error, we find that $i' = 16.1\%$.

The entry in the last year is shown to be $30,000 because the machinery will have this estimated MV. The asset, however, was depreciated to zero with the GDS method. Therefore, when the machine is sold at the end of year 10, there will be $30,000 of *recaptured depreciation*, or gain on disposal [Equation (7-16)], which is taxed at the effective income tax rate of 38%. This tax entry is shown in column *D* (EOY 10).

By trial and error, the after-tax IRR for Example 7-15 is found to be 12.4%.

(c) When MARR = 10% per year is inserted into the PW equation at the bottom of Table 7-6, it can be determined that the after-tax PW of this investment is $17,208.

Spreadsheet Solution

Figure 7-5 displays the spreadsheet solution for Example 7-15. This spreadsheet uses the form given in Figure 7-4 to compute the ATCFs. The spreadsheet also illustrates the use of the VDB (variable declining balance) function to compute MACRS (GDS) depreciation amounts.

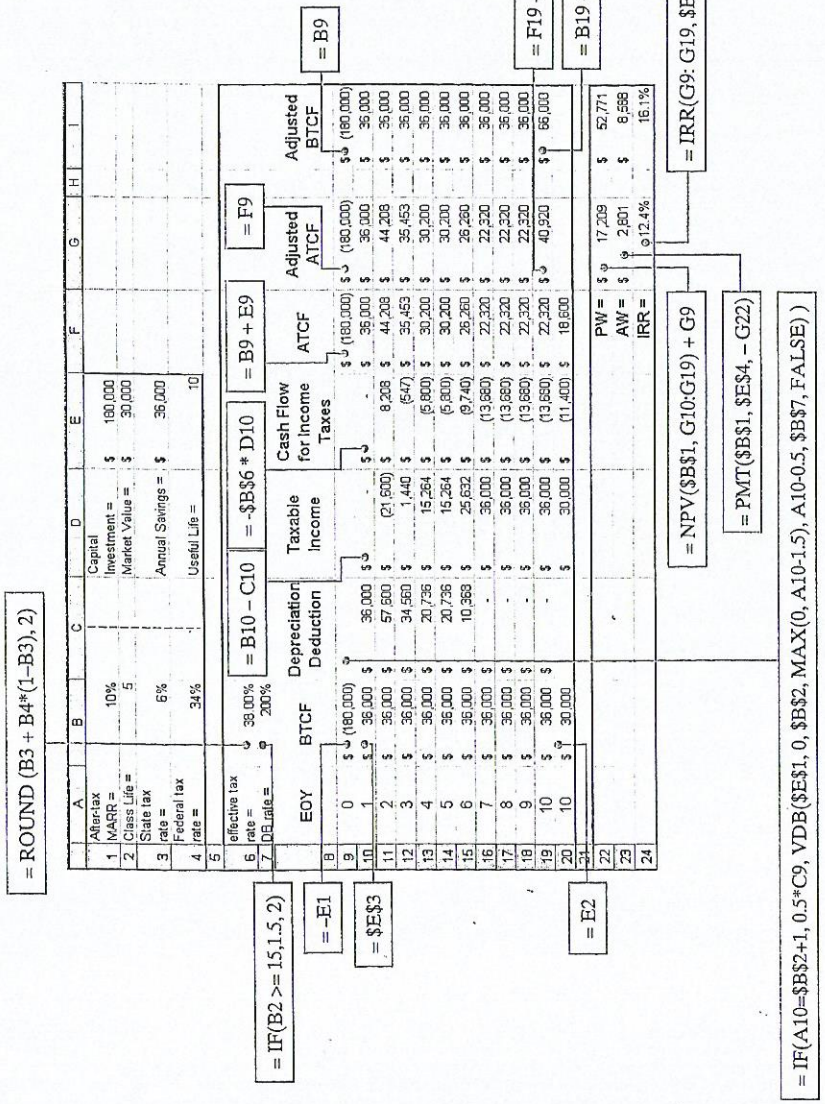
Cell B9 contains the cost basis, cells B10:B19 contain the BTCFs, and the year 10 MV is given in cell B20. The VDB function is used to determine the MACRS (GDS) depreciation amounts in column C. Cell B7 contains the DB percentage used for a five-year property class (see Table 7-4).

Note that in Figure 7-5, there are two row entries for the last year of the study period (year 10). The first entry accounts for expected revenues less expenses, while the second entry accounts for the disposal of the asset. To use the NPV and IRR financial functions, these values must be combined into a net cash flow for the year. This is accomplished by the adjusted ATCF and adjusted BTCF columns in the spreadsheet.

TABLE 7-6 ATCF Analysis of Example 7-15

End of Year, *k*	(A) BTCF	(B) Depreciation Deduction: Cost Basis	×	GDS Recovery Rate	=	Deduction	(C) = (A)−(B) Taxable Income	(D) = −0.38(C) Cash Flow for Income Taxes	(E) = (A) + (D) ATCF
0	−$180,000	—		—		—	0	0	−$180,000
1	36,000	$180,000	×	0.2000	=	$36,000	0	0	36,000
2	36,000	180,000	×	0.3200	=	57,600	−21,600	+8,208	44,208
3	36,000	180,000	×	0.1920	=	34,560	1,440	−547	35,453
4	36,000	180,000	×	0.1152	=	20,736	15,264	−5,800	30,200
5	36,000	180,000	×	0.1152	=	20,736	15,264	−5,800	30,200
6	36,000	180,000	×	0.0576	=	10,368	25,632	−9,740	26,260
7–10	36,000	0				0	36,000	−13,680	22,320
10	30,000						30,000[a]	−11,400[b]	18,600
Total	$210,000							Total	$130,201
								PW (10%) =	$17,208

[a] Depreciation recapture = $MV_{10} - BV_{10}$ = $30,000 − 0 = $30,000 (gain on disposal).
[b] Tax on depreciation recapture = $30,000(0.38) = $11,400.
After-tax IRR: Set PW of column E = 0 and solve for i' in the following equation:
$0 = -\$180{,}000 + \$36{,}000(P/F, i', 1) + \$44{,}208(P/F, i', 2) + \$35{,}453(P/F, i', 3) + \$30{,}200(P/F, i', 4) + \$30{,}200(P/F, i', 5) + \$26{,}260(P/F, i', 6) + \$22{,}320(P/A, i', 4)(P/F, i', 6) + \$18{,}600(P/F, i', 10)$; IRR = 12.4%.



	A	B	C	D	E	F	G	H	I
1	After-tax MARR =	10%		Capital Investment =	$ 180,000				
2	Class Life =	5		Market Value =	$ 30,000				
3	State tax rate =	6%		Annual Savings =	$ 36,000				
4	Federal tax rate =	34%		Useful Life =	10				
5									
6	effective tax rate =	38.00%							
7	DB rate =	200%							
8	EOY	BTCF	Depreciation Deduction	Taxable Income	Cash Flow for Income Taxes	ATCF	Adjusted ATCF		Adjusted BTCF
9	0	$ (180,000)				$ (180,000)	$ (180,000)		$ (180,000)
10	1	$ 36,000	$ 36,000	$ -	$ -	$ 36,000	$ 36,000		$ 36,000
11	2	$ 36,000	$ 57,600	$ (21,600)	$ 8,208	$ 44,208	$ 44,208		$ 36,000
12	3	$ 36,000	$ 34,560	$ 1,440	$ (547)	$ 35,453	$ 35,453		$ 36,000
13	4	$ 36,000	$ 20,736	$ 15,264	$ (5,800)	$ 30,200	$ 30,200		$ 36,000
14	5	$ 36,000	$ 20,736	$ 15,264	$ (5,800)	$ 30,200	$ 30,200		$ 36,000
15	6	$ 36,000	$ 10,368	$ 25,632	$ (9,740)	$ 26,260	$ 26,260		$ 36,000
16	7	$ 36,000	$ -	$ 36,000	$ (13,680)	$ 22,320	$ 22,320		$ 36,000
17	8	$ 36,000	$ -	$ 36,000	$ (13,680)	$ 22,320	$ 22,320		$ 36,000
18	9	$ 36,000	$ -	$ 36,000	$ (13,680)	$ 22,320	$ 22,320		$ 36,000
19	10	$ 36,000	$ -	$ 36,000	$ (13,680)	$ 22,320	$ 40,920		$ 66,000
20	10	$ 30,000		$ 30,000	$ (11,400)	$ 18,600			
21									
22						PW =	$ 17,209		$ 52,771
23						AW =	$ 2,801		$ 8,588
24						IRR =	12.4%		16.1%

Figure 7-5 Spreadsheet Solution, Example 7-15

In Example 7-16, we demonstrate the impact of a longer depreciation schedule on the after-tax profitability of an investment.

EXAMPLE 7-16

Impact of a Longer Depreciation Schedule on After-Tax PW and IRR

Suppose the machinery in Example 7-15 had been classified in the 10-year MACRS (GDS) property class. Calculate the new after-tax PW and after-tax IRR. Why are these results different than the results of Example 7-15?

Solution

If the machinery in Example 7-15 had been classified in the 10-year MACRS (GDS) property class instead of the five-year property class, depreciation deductions would be slowed down in the early years of the study period and shifted into later years, as shown in Table 7-7. Compared with entries in Table 7-6, entries in columns *C*, *D*, and *E* of Table 7-7 are less favorable, in the sense that a fair amount of ATCF is deferred until later years, producing a lower after-tax PW and IRR. For instance, the PW is reduced from $17,208 in Table 7-6 to $9,136 in Table 7-7. The basic difference between Table 7-6 and Table 7-7 is the *timing of the ATCF*, which is a function of the timing and magnitude of the depreciation deductions. In fact, the curious reader can confirm that the sums of entries in columns *A* through *E* of Tables 7-6 and 7-7 are nearly the same (except for the half-year of depreciation only in year 10 of Table 7-7). The timing of cash flows does, of course, make a difference!

Depreciation does not affect BTCF. Fast (accelerated) depreciation produces a larger PW of tax savings than does the same amount of depreciation claimed later in an asset's life.

A minor complication is introduced in ATCF analyses when the study period is shorter than an asset's MACRS recovery period (e.g., for a five-year recovery period, the study period is five years or less). In such cases, we shall assume throughout this book that the asset is sold for its MV in the last year of the study period. Due to the half-year convention, only one-half of the normal MACRS depreciation can be claimed in the year of disposal or end of the study period, so there will usually be a difference between an asset's BV and its MV. Resulting income tax adjustments will be made at the time of the sale (see the last row in Table 7-7) unless the asset in question is not sold but instead kept for standby service. In such a case, depreciation deductions usually continue through the end of the asset's MACRS recovery period. Our assumption of project termination at the end of the study period makes good economic sense, as illustrated in Example 7-17.

ATCF Analysis of Example 7-16 [Reworked Example 7-15 with Machinery in the 10-Year MACRS (GDS) Property Class]

End of Year, k	(A) BTCF	(B) Depreciation Deduction: Cost Basis	× GDS Recovery Rate	= Deduction	(C) = (A) − (B) Taxable Income	(D) = −0.38(C) Cash Flow for Income Taxes	(E) = (A) + (D) ATCF
0	−$180,000	—	—	—			−$180,000
1	36,000	$180,000 ×	0.1000 =	$18,000	$18,000	−$6,840	29,160
2	36,000	180,000 ×	0.1800 =	32,400	3,600	−1,368	34,632
3	36,000	180,000 ×	0.1440 =	25,920	10,080	−3,830	32,170
4	36,000	180,000 ×	0.1152 =	20,736	15,264	−5,800	30,200
5	36,000	180,000 ×	0.0922 =	16,596	19,404	−7,374	28,626
6	36,000	180,000 ×	0.0737 =	13,266	22,734	−8,639	27,361
7	36,000	180,000 ×	0.0655 =	11,790	24,210	−9,200	26,800
8	36,000	180,000 ×	0.0655 =	11,790	24,210	−9,200	26,800
9	36,000	180,000 ×	0.0656 =	11,808	24,192	−9,193	26,807
10	36,000	180,000 ×	0.0655/2 =	5,895	30,105	−11,440	24,560
10	30,000				18,201[a]	−6,916	23,084
						Total	$130,196
						PW (10%) ≃	$9,136
						IRR =	11.2%

[a] Gain on disposal $= MV_{10} - BV_{10} = \$30,000 - \left(\frac{0.0655}{2} + 0.0328\right)(\$180,000) = \$18,201$.

EXAMPLE 7-17

Study Period < MACRS Recovery Period

A highly specialized piece of equipment has a first cost of $50,000. If this equipment is purchased, it will be used to produce income (through rental) of $20,000 per year for only four years. At the end of year four, the equipment will be sold for a negligible amount. Estimated annual expenses for upkeep are $3,000 during each of the four years. The MACRS (GDS) recovery period for the equipment is seven years, and the firm's effective income tax rate is 40%.

(a) If the after-tax MARR is 7% per year, should the equipment be purchased?

(b) Rework the problem, assuming that the equipment is placed on standby status such that depreciation is taken over the full MACRS recovery period.

Solution

(a)

End of Year, k	(A) BTCF	(B) Depreciation Deduction	(C) = (A) − (B) Taxable Income	(D) = −0.4(C) Cash Flow for Income Taxes	(E) = (A) + (D) ATCF
0	−$50,000				−$50,000
1	17,000	$7,145	$9,855	−$3,942	13,058
2	17,000	12,245	4,755	−1,902	15,098
3	17,000	8,745	8,255	−3,302	13,698
4	17,000	3,123[a]	13,877	−5,551	11,449
4	0		−18,742[b]	7,497	7,497

[a] Half-year convention applies with disposal in year four.
[b] Remaining BV.

PW(7%) = $1,026. Because the PW > 0, the equipment should be purchased.

(b)

End of Year, k	(A) BTCF	(B) Depreciation Deduction	(C) Taxable Income	(D) Cash Flow for Income Taxes	(E) ATCF
0	−$50,000				−$50,000
1	17,000	$7,145	$9,855	−$3,942	13,058
2	17,000	12,245	4,755	−1,902	15,098
3	17,000	8,745	8,255	−3,302	13,698
4	17,000	6,245	10,755	−4,302	12,698
5	0	4,465	−4,465	1,786	1,786
6	0	4,460	−4,460	1,784	1,784
7	0	4,465	−4,465	1,786	1,786
8	0	2,230	−2,230	892	892
8	0				0

PW(7%) = $353, so the equipment should be purchased.

The PW is $673 higher in Part (a), which equals the PW of deferred depreciation deductions in Part (b). A firm would select the situation in Part (a) if it had a choice.

An illustration of determining ATCFs for a somewhat more complex, though realistic, capital investment opportunity is provided in Example 7-18.

EXAMPLE 7-18

After-Tax Analysis of an Integrated Circuit Production Line

The Ajax Semiconductor Company is attempting to evaluate the profitability of adding another integrated circuit production line to its present operations. The company would need to purchase two or more acres of land for $275,000 (total). The facility would cost $60,000,000 and have no net MV at the end of five years. The facility could be depreciated using a GDS recovery period of five years. An increment of working capital would be required, and its estimated amount is $10,000,000. Gross income is expected to increase by $30,000,000 per year for five years, and operating expenses are estimated to be $8,000,000 per year for five years. The firm's effective income tax rate is 40%.

(a) Set up a table and determine the ATCF for this project.

(b) Is the investment worthwhile when the after-tax MARR is 12% per year?

Solution

After-Tax Analysis of Example 7-18

End of Year, *k*	(*A*) BTCF	(*B*) Depreciation Deduction	(*C*) = (*A*)−(*B*) Taxable Income	(*D*) = −0.4(*C*) Cash Flow for Income Taxes	(*E*) = (*A*) + (*D*) ATCF
0	−$60,000,000 −10,000,000 −275,000				−$70,275,000
1	22,000,000	$12,000,000	$10,000,000	−$4,000,000	18,000,000
2	22,000,000	19,200,000	2,800,000	−1,120,000	20,880,000
3	22,000,000	11,520,000	10,480,000	−4,192,000	17,808,000
4	22,000,000	6,912,000	15,088,000	−6,035,200	15,964,800
5	22,000,000	3,456,000	18,544,000	−7,417,600	14,582,400
5	10,275,000[a]		−6,912,000[b]	2,764,800[b]	13,039,800

[a] MV of working capital and land.
[b] Because BV_5 of the production facility is $6,912,000 and net $MV_5 = 0$, a loss on disposal would be taken at EOY 5.

(a) The format recommended in Figure 7-4 is followed in Table 7-8 to obtain ATCFs in years zero through five. Acquisitions of land, as well as additional working capital, are treated as nondepreciable capital investments whose MVs at the end of year five are estimated to equal their first costs. (In economic evaluations, it is customary to assume that land and working capital do not inflate in value during the study period because they are "nonwasting" assets.) By using a variation of Equation (7-21), we are able to compute ATCF in year three (for example) to be

$$\text{ATCF}_3 = (\$30{,}000{,}000 - \$8{,}000{,}000 - \$11{,}520{,}000)(1 - 0.40) + \$11{,}520{,}000$$
$$= \$17{,}808{,}000.$$

(b) The depreciable property in Example 7-18 ($60,000,000) will be disposed of for $0 at the end of year five, and a loss on disposal of $6,912,000 will be claimed at the end of year five. Only a half-year of depreciation ($3,456,000) can be claimed as a deduction in year five, and the BV is $6,912,900 at the end of year five. Because the selling price (MV) is zero, the loss on disposal equals our BV of $6,912,000. As seen from Figure 7-4, a tax credit of 0.40($6,912,000) = $2,764,800 is created at the end of year five. The after-tax IRR is obtained from entries in column E of Table 7-8 and is found to be 12.5%. The after-tax PW equals $936,715 at MARR = 12% per year. Based on economic considerations, this integrated circuit production line should be recommended because it appears to be quite attractive.

In the next example, the after-tax comparison of mutually exclusive alternatives involving only costs is illustrated.

EXAMPLE 7-19

After-Tax Comparison of Purchase versus Leasing Alternatives

An engineering consulting firm can purchase a fully configured Computer-Aided Design (CAD) workstation for $20,000. It is estimated that the useful life of the workstation is seven years, and its MV in seven years should be $2,000. Operating expenses are estimated to be $40 per eight-hour workday, and maintenance will be performed under contract for $8,000 per year. The MACRS (GDS) property class is five years, and the effective income tax rate is 40%.

As an alternative, sufficient computer time can be leased from a service company at an annual cost of $20,000. If the after-tax MARR is 10% per year, how many workdays per year must the workstation be needed in order to justify *leasing* it?

Solution

This example involves an after-tax evaluation of purchasing depreciable property versus leasing it. We are to determine how much the workstation must be utilized so that the lease option is a good economic choice. A *key* assumption is that the cost of engineering design time (i.e., operator time) is unaffected by whether the workstation is purchased or leased. Variable operations expenses associated with ownership result from the purchase of supplies, utilities, and so on. Hardware and software maintenance cost is contractually fixed at $8,000 per year. It is further assumed that the maximum number of working days per year is 250.

Lease fees are treated as an annual expense, and the consulting firm (the lessee) may *not* claim depreciation of the equipment to be an additional expense. (The leasing company presumably has included the cost of depreciation in its fee.) Determination of ATCF for the lease option is relatively straightforward and is not affected by how much the workstation is utilized:

$$(\text{After-tax expense of the lease})_k = -\$20{,}000(1-0.40) = -\$12{,}000;\ k = 1, \ldots, 7.$$

ATCFs for the purchase option involve expenses that are fixed (not a function of equipment utilization) in addition to expenses that vary with equipment usage. If we let X equal the number of working days per year that the equipment is utilized, the variable cost per year of operating the workstation is $40X. The after-tax analysis of the purchase alternative is shown in Table 7-9.

The after-tax annual worth (AW) of purchasing the workstation is

$$\begin{aligned}\text{AW}(10\%) &= -\$20{,}000(A/P, 10\%, 7) - \$24X - [\$3{,}200(P/F, 10\%, 1) + \cdots \\ &\quad + \$4{,}800(P/F, 10\%, 7)](A/P, 10\%, 7) + \$1{,}200(A/F, 10\%, 7) \\ &= -\$24X - \$7{,}511.\end{aligned}$$

TABLE 7-9 After-Tax Analysis of Purchase Alternative (Example 7-19)

End of Year, k	(A) BTCF	(B) Depreciation Deduction[a]	(C) = (A) − (B) Taxable Income	(D) = −t(C) Cash Flow for Income Taxes	(E) = (A) + (D) ATCF
0	−$20,000				−$20,000
1	−40X − 8,000	$4,000	−$40X − $12,000	$16X + $4,800	−24X − 3,200
2	−40X − 8,000	6,400	−40X − 14,400	16X + 5,760	−24X − 2,240
3	−40X − 8,000	3,840	−40X − 11,840	16X + 4,736	−24X − 3,264
4	−40X − 8,000	2,304	−40X − 10,304	16X + 4,122	−24X − 3,878
5	−40X − 8,000	2,304	−40X − 10,304	16X + 4,122	−24X − 3,878
6	−40X − 8,000	1,152	−40X − 9,152	16X + 3,661	−24X − 4,339
7	−40X − 8,000	0	−40X − 8,000	16X + 3,200	−24X − 4,800
7	2,000		2,000	−800	1,200

[a] Depreciation deduction$_k$ = $20,000 × (GDS recovery rate).

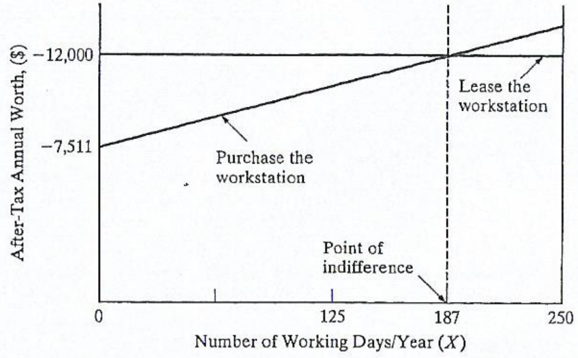


Figure 7-6 Summary of Example 7-19

To solve for X, we equate the after-tax annual worth of both alternatives:

$$-\$12{,}000 = -\$24X - \$7{,}511.$$

Thus, X = 187 days per year. Therefore, if the firm expects to utilize the CAD workstation in its business *more than* 187 days per year, the equipment should be leased. The graphic summary of Example 7-19 shown in Figure 7-6 provides the rationale for this recommendation. The importance of the workstation's estimated utilization, in workdays per year, is now quite apparent.

EXAMPLE 7-20

After-Tax Analysis of Alternatives with Unequal Lives

A firm must decide between two system designs, *S1* and *S2*, whose estimated cash flows are shown in the following table. The effective income tax rate is 40% and MACRS (GDS) depreciation is used. Both designs have a GDS recovery period of five years. If the after-tax desired return on investment is 10% per year, which design should be chosen?

	Design	
	S1	*S2*
Capital investment	$100,000	$200,000
Useful life (years)	7	6
MV at end of useful life	$30,000	$60,000
Annual revenues less expenses	$20,000	$40,000

Solution

Note that the design alternatives have different useful lives. The same basic principles of engineering economy apply to both before-tax and after-tax analyses. Therefore, we must analyze the two system designs over a common period of time. As we discovered in Chapter 6, using the repeatability assumption along with the annual worth method simplifies the analysis of alternatives having unequal lives.

Both alternatives would be depreciated using a five-year GDS recovery period. No adjustments to the GDS rates are required because the useful life of each alternative is greater than or equal to six years of depreciation deductions. Tables 7-10 and 7-11 summarize the calculation of the ATCFs for the design alternatives.

After-Tax Analysis of Design *S1*, Example 7-20

End of Year, k	(*A*) BTCF	(*B*) Depreciation Deduction	(*C*) = (*A*) − (*B*) Taxable Income	(*D*) = − *t*(*C*) Cash Flow for Income Taxes	(*E*) = (*A*) + (*D*) ATCF	PW(10%)
0	−$100,000				−$100,000	−$100,000
1	20,000	$20,000	$0	$0	20,000	18,182
2	20,000	32,000	−12,000	4,800	24,800	20,495
3	20,000	19,200	800	−320	19,680	14,786
4	20,000	11,520	8,480	−3,392	16,608	11,343
5	20,000	11,520	8,480	−3,392	16,608	10,312
6	20,000	5,760	14,240	−5,696	14,304	8,075
7	20,000	0	20,000	−8,000	12,000	6,158
7	30,000		30,000	−12,000	18,000	9,238
						$PW_{S1}(10\%) = -\$1{,}411$

After-Tax Analysis of Design *S2*, Example 7-20

End of Year, k	(*A*) BTCF	(*B*) Depreciation Deduction	(*C*) = (*A*) − (*B*) Taxable Income	(*D*) = −*t*(*C*) Cash Flow for Income Taxes	(*E*) = (*A*) + (*D*) ATCF	PW(10%)
0	−$200,000				−$200,000	−$200,000
1	40,000	$40,000	$0	$0	40,000	36,364
2	40,000	64,000	−24,000	9,600	49,600	40,989
3	40,000	38,400	1,600	−640	39,360	29,571
4	40,000	23,040	16,960	−6,784	33,216	22,687
5	40,000	23,040	16,960	−6,784	33,216	20,624
6	40,000	11,520	28,480	−11,392	28,608	16,149
6	50,000		50,000	−20,000	30,000	16,935
						$PW_{S2}(10\%) = -\$16{,}681$

We can't directly compare the PW of the after-tax cash flows because of the difference in the lives of the alternatives. We can, however, directly compare the AWs of the ATCFs by using the repeatability assumption from Chapter 6.

$$AW_{S1}(10\%) = PW_{S1}(A/P, 10\%, 7) = -\$1{,}411(0.2054) = -\$290$$

$$AW_{S2}(10\%) = PW_{S2}(A/P, 10\%, 6) = -\$16{,}681(0.2296) = -\$3{,}830$$

Based on an after-tax annual worth analysis, Design *S1* is preferred since it has the greater (less negative) AW. Neither design however makes money, so if a system is not required, don't recommend either one.

7.11 Economic Value Added

This section discusses an economic measure for estimating the wealth creation potential of capital investments that is experiencing increased attention and use. The measure, called economic value added (EVA),* can be determined from some of the data available in an after-tax analysis of cash flows generated by a capital investment. Through retroactive analysis of a firm's common stock valuation, it has been established that some companies experience a statistically significant relationship between the EVA metric and the historical value of their common stock.† For our purposes, EVA can also be used to estimate the profit-earning *potential* of proposed capital investments in engineering projects.

Simply stated, EVA is the difference between the company's adjusted net operating profit after taxes (NOPAT) in a particular year and its after-tax cost of capital during that year. Another way to characterize EVA is "the spread between the return on the capital and the cost of the capital."‡ On a project-by-project basis (i.e., for discrete investments), the EVA metric can be used to gauge the wealth creation opportunity of proposed capital expenditures. We now define annual EVA as

$$\begin{aligned} EVA_k &= (\text{Net operating profit after taxes})_k \\ &\quad - (\text{Cost of capital used to produce profit})_k \\ &= NOPAT_k - i \cdot BV_{k-1}, \end{aligned} \qquad (7\text{-}22)$$

where

k = an index for the year in question($1 \le k \le N$);

i = after-tax MARR based on a firm's cost of capital,

BV_{k-1} = beginning-of-year book value;

N = the study (analysis) period in years.

* EVA is a registered trademark of Stern Stewart & Company, New York City, NY.

† See J. L. Dodd and S. Chen, "EVA: A New Panacea?" *B & E Review*, 42 (July–September 1996): 26–28, and W. Freedman, "How Do You Add Up?" *Chemical Week*, October 9, 1996, pp. 31–34.

‡ S. Tully, "The Real Key To Creating Wealth," *Fortune*, September 30, 1993, p. 38ff.

$NOPAT_k$ can be determined from Figure 7-4. It is simply the algebraic addition of the entry in Column C and the entry in Column D.

$$NOPAT_k = \text{Taxable income} + \text{Cash flow for income taxes}$$
$$= (R_k - E_k - d_k) + (-t)(R_k - E_k - d_k)$$
$$= (R_k - E_k - d_k)(1 - t) \quad (7\text{-}23)$$

Substituting Equation (7-21) into Equation (7-23), we can see the relationship between $ATCF_k$ and $NOPAT_k$ to be

$$NOPAT_k = ATCF_k - d_k. \quad (7\text{-}24)$$

Equation (7-23) and Figure 7-4 are demonstrated in Example 7-21 to determine the ATCF amounts, after-tax AW, and the EVA amounts related to a capital investment.

EXAMPLE 7-21 EVA

Consider the following proposed capital investment in an engineering project and determine its

(a) year-by-year ATCF,
(b) after-tax AW,
(c) annual equivalent EVA.

Proposed capital investment	= $84,000
Salvage value (end of year four)	= $0
Annual expenses per year	= $30,000
Gross revenues per year	= $70,000
Depreciation method	= Straight Line
Useful life	= four years
Effective income tax rate (t)	= 50%
After-tax MARR (i)	= 12% per year

Solution

(a) Year-by-year ATCF amounts are shown in the following table:

EOY	BTCF	Depreciation	Taxable Income	Income Taxes	ATCF
0	−$84,000	—	—	—	−$84,000
1	70,000 − 30,000	$21,000	$19,000	−$9,500	30,500
2	70,000 − 30,000	21,000	19,000	−9,500	30,500
3	70,000 − 30,000	21,000	19,000	−9,500	30,500
4	70,000 − 30,000	21,000	19,000	−9,500	30,500

(b) The annual equivalent worth of the ATCFs equals $-\$84{,}000(A/P, 12\%, 4) + \$30{,}500 = \$2{,}844$.

(c) The EVA in year k equals $NOPAT_k - 0.12\ BV_{k-1}$ [Equation (7-22)]. The year-by-year EVA amounts and the annual equivalent worth of EVA ($2,844) are shown in the next table. Hence, the after-tax AW and the annual equivalent worth of EVA of the project are *identical*.

EOY_k	NOPAT	$EVA = NOPAT - i \cdot BV_{k-1}$
1	$19,000 − $9,500 = $9,500	$9,500 − 0.12($84,000) = −$580
2	= $9,500	$9,500 − 0.12($63,000) = $1,940
3	= $9,500	$9,500 − 0.12($42,000) = $4,460
4	= $9,500	$9,500 − 0.12($21,000) = $6,980

Annual equivalent EVA $= [-\$580(P/F, 12\%, 1) + \$1{,}940(P/F, 12\%, 2) + \$4{,}460(P/F, 12\%, 3) + \$6{,}980(P/F, 12\%, 4)](A/P, 12\%, 4) = \$2{,}844$.

In Example 7-21, it was shown that the after-tax AW (12%) of the proposed engineering project is identical to the annual equivalent EVA at the same interest rate. Therefore, the annual equivalent EVA is simply the annual worth, at the after-tax MARR, of a project's ATCFs. This straightforward relationship is also valid when accelerated depreciation methods (such as MACRS) are used in the analysis of a proposed project. The reader is referred to Problems 7-47, 7-48, and 7-49 at the end of the chapter for EVA exercises.

7.12 Summary

In this chapter, we have presented important aspects of federal legislation relating to depreciation and income taxes. It is essential to understand these topics so that correct after-tax engineering economy evaluations of proposed projects may be conducted. Depreciation and income taxes are also integral parts of subsequent chapters in this book.

In this chapter, many concepts regarding current federal income tax laws were described. For example, topics such as taxable income, effective income tax rates, taxation of ordinary income, and gains and losses on disposal of assets were explained. A general format for pulling together and organizing all these apparently diverse subjects was presented in Figure 7-4. This format offers the student or practicing engineer a means of collecting, on one worksheet, information that is required for determining ATCFs and properly evaluating the after-tax economic worth of a proposed capital investment. Figure 7-4 was then employed in numerous examples. The student's challenge is now to use this worksheet in organizing data presented in problem exercises at the end of this and subsequent chapters and to answer questions regarding the after-tax profitability.

Problems

The number in parentheses that follows each problem refers to the section from which the problem is taken.

7-1. How are depreciation deductions different from other production or service expenses such as labor, material, and electricity? **(7.2)**

7-2. What conditions must a property satisfy to be considered depreciable? **(7.2)**

7-3. Explain the difference between real and personal property. **(7.2)**

7-4. Explain how the cost basis of depreciable property is determined. **(7.2)**

7-5. Why would a business elect, under MACRS, to use the ADS rather than the GDS? **(7.4)**

7-6. The "Big-Deal" Company has purchased new furniture for their offices at a retail price of $100,000. An additional $20,000 has been charged for insurance, shipping, and handling. The company expects to use the furniture for 10 years (useful life = 10 years) and then sell it at a salvage (market) value of $10,000. Use the SL method for depreciation to answer these questions. **(7.3)**

a. What is the depreciation during the second year?

b. What is the BV of the asset at the end of the first year?

c. What is the BV of the asset after 10 years?

7-7. Cisco Systems is purchasing a new bar code–scanning device for its service center in San Francisco. The table that follows lists the relevant cost items for this purchase. The operating expenses for the new system are $10,000 per year, and the useful life of the system is expected to be five years. The SV for depreciation purposes is equal to 25% of the hardware cost. **(7.3)**

Cost Item	Cost
Hardware	$160,000
Training	$15,000
Installation	$15,000

a. What is the BV of the device at the end of year three if the SL depreciation method is used?

b. Suppose that after depreciating the device for two years with the SL method, the firm decides to switch to the double declining balance depreciation method for the remainder of the device's life (the remaining three years). What is the device's BV at the end of four years?

7-8. An asset for drilling was purchased and placed in service by a petroleum production company. Its cost basis is $60,000, and it has an estimated MV of $12,000 at the end of an estimated useful life of 14 years. Compute the depreciation amount in the third year and the BV at the end of the fifth year of life by each of these methods: **(7.3, 7.4)**

a. The SL method.

b. The 200% DB method with switchover to SL.

c. The GDS.

d. The ADS.

7-9. A cupola for a foundry was purchased for $3,000. $500 more was spent on its erection and commissioning. The estimated residual value after 10 years was $700. **(7.3)**

a. Calculate the annual rate of depreciation.

b. Determine the amount of depreciation at the end of six years after the purchase of the cupola.

7-10. An asset was purchased for $5,000. After the SL method was used to calculate depreciation for a total life of 25 years, the asset's expected salvage value is $500. What would the difference be between the asset's book value (after 15 years) and the book value that would have resulted if the the DB method at the rate of 10% applied for 15 years had been used instead? **(7.1)**

7-11. Your company has purchased a large new truck-tractor for over-the-road use (asset class 00.26). It has a cost basis of $180,000. With additional options costing $15,000, the cost basis for depreciation purposes is $195,000. Its MV at the end of five years is estimated as $40,000. Assume it will be depreciated under the GDS: **(7.4)**

a. What is the cumulative depreciation through the end of year three?

b. What is the MACRS depreciation in the fourth year?

c. What is the BV at the end of year two?

7-12. A construction company is considering changing its depreciation from the MACRS method to the historical SL method for a general purpose hauling truck. The cost basis of the truck is $100,000, and the expected salvage value for depreciation purposes is $8,000. The company will use the truck for eight years and will depreciate it over this period of time with the SL method. What is the difference in the amount of depreciation that would be claimed in year five (i.e., MACRS versus SL)? **(7.3, 7.4)**

7-13. A CNC milling machine has a cost basis of $40,000 and a depreciable life of 10 years. The estimated salvage value of the machine is zero at the end of the 10-year period.

Use the DB method to calculate the annual depreciation amounts when **(7.1, 7.2)**

a. $R = 2/N$

b. $R = 1.5/N$

c. SL

7-14. During its current tax year (year one), a pharmaceutical company purchased a mixing tank that had a fair market price of $120,000. It replaced an older, smaller mixing tank that had a BV of $15,000. Because a special promotion was underway, the old tank was used as a trade-in for the new one, and the cash price (including delivery and installation) was set at $99,500. The MACRS class life for the new mixing tank is 9.5 years. **(7.4, 7.3)**

a. Under the GDS, what is the depreciation deduction in year three?

b. Under the GDS, what is the BV at the end of year four?

c. If 200% DB depreciation had been applied to this problem, what would be the cumulative depreciation through the end of year four?

7-15. Firm ABC bought a piece of equipment 6 years ago, which firm XYZ has just purchased for $1,500,000. The equipment is expected to be operational for 20 more years. If its class life is 12 years, what property class does the equipment belong to under MACRS (GDS)? What is the depreciation charged for the equipment 5 years after firm XYZ's purchase? Also find the BV at the end of 5th year if the equipment is disposed of then. **(7.4)**

7-16. In June 2008, your company traded-in a used car with a BV of $5,000. The vendor, having accepted the older car as a trade-in, was willing to make a deal and offered to sell you a new car with an MV of $40,000 for $35,000 in cash. Find

a. What GDS property class the new car would fit into?

b. How much depreciation can be deducted each year based on this class life? **(7.5)**

7-17. A concrete and rock crusher for demolition work has been purchased for $60,000, and it has an estimated SV of $10,000 at the end of its five-year life. Engineers have estimated that the following units of production (in m^3 of crushed material) will be contracted over the next five years.

EOY	1	2	3	4	5
m^3	16,000	24,000	36,000	16,000	8,000

Using the units of production depreciation method, what is the depreciation allowance in year three, and what is the BV at the end of year two? **(7.3)**

7-18. During a particular year, a corporation has $18.6 million in revenue, $2.4 million of operating expenses, and depreciation expenses of $6.4 million. What is the approximate federal tax this corporation will have to pay for this tax year? **(7.7)**

7-19. Electrostatic precipitators costing $2,000 are used to separate ash particles from hot flue gases being discharged into the air. These precipitators must be replaced every 3,000 hours. If the plant operates 24 hours a day and 30 days a month, what is the depreciation charged per month for the filters? **(7.1)**

7-20. Office furniture and equipment have been purchased by a laboratory for $45,000. The expected service life is 9 years and projected salvage value is $7,000. Show each year's depreciation charged under MACRS (GDS) over the recovery period for that type of property. **(7.4)**

7-21. A power producing unit has installed a new computer to control one of its units. For a 5-year property class, what would the depreciation percentage be for the third year after the installation when using MACRS (GDS)? **(7.4)**

7-22. A $125,000 tractor-trailer is being depreciated by the SL method over five years to a final BV of zero. Half-year convention does not apply to this asset. After three years, the rig is sold for (a) $70,000 or (b) $20,000. If the effective income tax rate is 40%, what is the net cash inflow from the sale for situation (a) and situation (b)? **(7.8)**

7-23. A new machine costs $200,000 and is expected to reduce net annual operating costs by $40,000 per year for 10 years. It will have an MV of $40,000 at the end of the tenth year. **(7.6)**

a. Find ATCFs and BTCFs with recovery period of 5 years.

b. Calculate before-tax and after-tax IRR.

c. Calculate present worth of the cash flow after tax (MARR = 10% per year).

Assume effective income tax rate is 40%.

7-24. Refer to Example 6-10. Work this problem on an after-tax basis when the MARR is 12% per year. The effective income tax rate is 40%, and MACRS depreciation is appropriate with a property class of five years. Recall that the market values of *M1* and *M2* are zero at the end of years five and eight, respectively. **(7.9)**

7-25. A semiconductor manufacturer is considering the installation of an automatic control system. This is estimated to produce savings in the production process equivalent to $40,000 per year. The equipment has an initial cost of $50,000 and an estimated operating cost of $20,000. Salvage value at the end of its economic life of 7 years will be $5,000. Assume that this control system is depreciated as MACRS (GDS) five-year property class life, that the effective income tax is 40% and that capital gains are to be considered (fixed at 35%). Find the PW of ATCFs if MARR is 15%. **(7.14)**

7-26. An assembly operation at a software company currently requires $100,000 per year in labor costs. A robot can be purchased and installed to automate this operation, and the robot will cost $200,000 with no MV at the end of its 10-year life. The robot, if acquired, will be depreciated using SL depreciation to a terminal BV of zero after 10 years. Maintenance and operation expenses of the robot are estimated to be $64,000 per year. The company has an effective income tax rate of 40%. Invested capital must earn at least 8% after income taxes are taken into account. **(7.9)**

a. Use the IRR method to determine if the robot is a justifiable investment.

b. If MACRS (seven-year recovery period) had been used in Part (a), would the after-tax IRR be lower or higher than your answer to Part (a)?

7-27. Liberty Airways is considering an investment of $800,000 in ticket purchasing kiosks at selected airports. The kiosks (hardware and software) have an expected life of four years. Extra ticket sales are expected to be 60,000 per year at a discount price of $40 per ticket. Fixed costs, excluding depreciation of the equipment, are $400,000 per year, and variable costs are $24 per ticket. The kiosks will be depreciated over four years, using the SL method with a zero salvage value. The onetime commitment of working capital is expected to be 1/12 of annual sales dollars. The after-tax MARR is 15% per year, and the company pays income tax at the rate of 34%. What's the after-tax PW of this proposed investment? Should the investment be made? **(7.9)**

7-28. Nordique Fab is an Arizona company dedicated to circuit board design and fabrication. It has just acquired new workstations and modeling software for its three "Valley of the Sun" design facilities, at a cost of $425,000 per site. This cost includes the hardware, software, transportation, and installation costs. Additional software training has been purchased at a cost of $25,000 per site. The estimated MV for each system during the fourth year is expected to be 5% of the total capital investment, at which time the systems will all be sold. The company believes that use of the new systems will enhance their circuit design business, resulting in a total increase in annual income of $1,000,000. The engineering design manager wants to determine the tax implications of this purchase. He estimates that annual operating and maintenance costs on the systems will be approximately $220,000 (all sites combined). The company's marginal effective tax rate is 35% and the MACRS depreciation method (with a five-year GDS recovery period) will be used. Use Figure 7-4 to determine the after-tax cash flows for this project. If the after-tax MARR is 20% per year, would you recommend this investment? **(7.9)**

7-29. Your company has just signed a three-year nonrenewable contract with the city of New Orleans for earthmoving work. You are investigating the purchase of heavy construction equipment for this job. The equipment costs $200,000 and qualifies for five-year MACRS depreciation. At the end of the three-year contract, you expect to be able to sell the equipment for $70,000. If the projected operating expense for the equipment is $65,000 per year, what is the after-tax equivalent uniform annual cost (EUAC) of owning and operating this equipment? The effective income tax rate is 40%, and the after-tax MARR is 12% per year. **(7.9)**

7-30. The Greentree Lumber Company is attempting to evaluate the profitability of adding another cutting line to its present sawmill operations. They would need to purchase two more acres of land for $30,000 (total). The equipment would cost $130,000 and could be depreciated over a five-year recovery period with the MACRS method. Gross revenue is expected to increase by $50,000 per year for five years, and operating expenses will be $15,000 annually for five years. It is expected that this cutting line will be closed down after five years. The firm's effective income tax rate is 50%. If the company's after-tax MARR is 5% per year, is this a profitable investment? **(7.9)**

7-31. A drug store is looking into the possibility of installing a "24/7" automated prescription refill system to increase its projected revenues by $20,000 per year over the next five years. Annual expenses to maintain the system are expected to be $5,000. The system will have no market value at the end of its five-year life, and it will be depreciated by the SL method. The store's effective income tax rate is 40%, and the after-tax MARR is 12% per year. What is the maximum amount that is justified for the purchase of this prescription refill system? **(7.9)**

7-32. Your company is considering the introduction of a new product line. The initial investment required for this project is $500,000, and annual maintenance costs are anticipated to be $35,000. Annual operating cost will be directly in proportion to the level of production at $7.50 per unit, and each unit of product can be sold for $50.00. If the project has a life of five years, what is the minimum annual production level for which this project is economically viable? Work this problem on an after-tax basis. Assume five-year SL depreciation ($SV_5 = 0$), an effective income tax rate of 40%, and an after-tax MARR of 10% per year. **(7.9)**

7-33. Your company has purchased equipment (for $50,000) that will reduce materials and labor costs by $14,000 each year for *N* years. After *N* years, there will be no further need for the machine, and because the machine is specially designed, it will have no MV at any time. The IRS, however, has ruled that you must depreciate the equipment on a SL basis with a tax life of five years. If the effective income tax rate is 40%, what is the minimum number of years your firm must operate the equipment to earn 10% per year after taxes on its investment? **(7.9)**

7-34. Refer to Problem 6-79. The alternatives all have a MACRS (GDS) property class of three years. If the effective income tax rate is 40% and the after-tax MARR = (1 − 0.4)(12%) = 7.2% per year, which alternative should be recommended? Is this the same recommendation you made when the alternatives were analyzed on a before-tax basis? **(7.10)**

7-35. The following information is for a proposed project that will provide the capability to produce a specialized product estimated to have a short market (sales) life:

- Capital investment is $1,000,000. (This includes land and working capital.)
- The cost of depreciable property, *which is part* of the $1,000,000 total estimated project cost, is $420,000.
- Assume, for simplicity, that the depreciable property is in the MACRS (GDS) three-year property class.
- The analysis period is three years.
- Annual operating and maintenance expenses are $636,000 in the first year, and they increase at the rate of 6% per year (i.e., $\bar{f}$ = 6%) thereafter. (See geometric gradient, Chapter 4.)
- Estimated MV of depreciable property from the project at the end of three years is $280,000.
- Federal income tax rate = 34%; state income tax rate = 4%.
- MARR (after taxes) is 10% per year.

Based on an after-tax analysis using the PW method, what minimum amount of equivalent uniform annual revenue is required to justify the project economically? **(7.9, 7.10)**

7-36. In a chlorine-fluxing installation in a large aluminum company, engineers are considering the replacement of existing plastic pipe fittings with more expensive, but longer lived, copper fittings. The following table gives a comparison of the capital investments, lives, salvage values, and so on of the two mutually exclusive alternatives under consideration:

	Plastic	Copper
Capital investment	$5,000	$10,000
Useful (class) life	5 years	10 years
Salvage value for depreciation purposes	$1,000(= SV_5)	$5,000(= SV_{10})
Annual expenses	$300	$100
Market value at end of useful life	$0	$0

Depreciation amounts are calculated with the SL method. Assume an income tax rate of 40% and a MARR after-taxes of 12% per year. Which pipe fitting would you select and why? Carefully list all assumptions that you make in performing the analysis. **(7.9, 7.10)**

7-37. An industrial coal-fired boiler for process steam is equipped with a 10-year-old electrostatic precipitator (ESP). Changes in coal quality have caused stack emissions to be in noncompliance with federal standards for particulates. Two mutually exclusive alternatives have been proposed to rectify this problem (doing nothing is not an option).

	New Baghouse	New ESP
Capital investment	$1,140,000	$992,500
Annual operating expenses	115,500	73,200

The life of both alternatives is 10 years, the effective income tax rate is 40%, and the after-tax MARR is 9% per year. Both alternatives qualify as seven-year MACRS (GDS) properties. Make a recommendation regarding which alternative to select based on an after-tax analysis. **(7.10)**

7-38. Storage tanks to hold a highly corrosive chemical are currently made of material Z26. The capital investment in a tank is $30,000, and its useful life is eight years. Your company manufactures electronic components and uses the ADS under MACRS to calculate depreciation deductions for these tanks. The net MV of the tanks at the end of their useful life is zero. When a tank is four years old, it must be relined at a cost of $10,000. This cost is not depreciated and can be claimed as an expense during year four.

Instead of purchasing the tanks, they can be leased. A contract for up to 20 years of storage tank service can be written with the Rent-All Company. If your firm's after-tax MARR is 12% per year, what is the greatest annual amount that you can afford to pay for tank leasing without causing purchasing to be the more economical alternative? Your firm's effective income tax rate is 40%. State any assumptions you make. **(7.4, 7.9)**

7-39. Two fixtures are being considered for a particular job in a manufacturing firm. The pertinent data for their comparison are summarized in Table P7-39.

The effective federal and state income tax rate is 50%. Depreciation recapture is also taxed at 50%. If the after-tax MARR is 8% per year, which of the two fixtures should be recommended? State any important assumptions you make in your analysis. **(7.9)**

7-40. Individual industries will use energy as efficiently as it is economical to do so, and there are several incentives to improve the efficiency of energy consumption. To illustrate, consider the selection of a new water pump. The pump is to operate 800 hours per year. Pump A costs $2,000, has an overall efficiency of 82.06%, and it delivers 11 hp. The other available alternative, pump B, costs $1,000, has an overall efficiency of 45.13%, and delivers 12.1 hp. Both pumps have a useful life of five years and will be sold at that time. (Remember 1 hp = 0.746 kW.)

Table for Problem 7-39

	Fixture X	Fixture Y
Capital investment	$30,000	$40,000
Annual operating expenses	$3,000	$2,500
Useful life	6 years	8 years
Market value	$6,000	$4,000
Depreciation method	SL to zero book value over 5 years	MACRS (GDS) with 5-year recovery period

Pump A will use SL depreciation over five years with an estimated SV of zero. Pump B will use the MACRS depreciation method with a class life of three years. After five years, pump A has an actual market value of $400, and pump B has an actual market value of $200.

Using the IRR method on the after-tax cash flows and a before-tax MARR of 16.67%, is the incremental investment in pump A economically justifiable? The effective income tax rate is 40%. The cost of electricity is $0.05/kWh, and the pumps are subject to a study period of five years. **(7.10)**

7-41. Two alternative machines will produce the same product, but one is capable of higher-quality work, which can be expected to return greater revenue. The following are relevant data:

	Machine A	Machine B
Capital investment	$20,000	$30,000
Life	12 years	8 years
Terminal BV (and MV)	$4,000	$0
Annual receipts	$150,000	$188,000
Annual expenses	$138,000	$170,000

Determine which is the better alternative, assuming repeatability and using SL depreciation, an income-tax rate of 40%, and an after-tax MARR of 10%. **(7.9)**

7-42. Two types of machine tools are available for performing a particular task in a manufacturing firm. The estimated cost and salvage values are given as follows:

	Machine A	Machine B
First cost	$40,000	$50,000
Operating cost per year	$4,000	$2,000
Service life	6 years	8 years
Salvage value	$5,000	$3,000
Depreciation method	MACRS (GDS)	MACRS (GDS)
Recovery period	5 years	5 years

The effective income tax rate is 40%. Make a comparison after income tax for both machines, using a 10% MARR. **(7.20)**

7-43. Alternative Methods *I* and *II* are proposed for a security operation. The following is comparative information:

	Method *I*	Method *II*
Initial investment	$10,000	$40,000
Useful (ADR) life	5 years	10 years
Terminal market value	$1,000	$5,000
Annual expenses		
Labor	$12,000	$4,000
Power	$250	$300
Rent	$1,000	$500
Maintenance	$500	$200
Property taxes and insurance	$400	$2,000
Total annual expenses	$14,150	$7,000

Determine which is the better alternative based on an after-tax annual cost analysis with an effective income tax of 40% and an after-tax MARR of 12%, assuming the following methods of depreciation: **(7.9)**

a. SL

b. MACRS

7-44. A firm is considering buying a new machine and has to choose between two options. The specifications of each are given below:

	Machine *I*	Machine *II*
Initial cost	$100,000	$80,000
Operating cost per year	$20,000	$25,000
Economic life	5 years	5 years
Final salvage value	$40,000	$15,000

For each machine, assume a three-year MACRS (GDS) recovery property and depreciation class, with an effective income tax rate of 50%, a before-tax MARR of 25% and after-tax MARR of 15%. Find the alternative that should be selected. **(7.20)**

7-45. A manufacturing process can be designed for varying degrees of automation. The following is relevant cost information:

Degree	First Cost	Annual Labor Expense	Annual Power and Maintenance Expense
A	$10,000	$9,000	$500
B	14,000	7,500	800
C	20,000	5,000	1,000
D	30,000	3,000	1,500

Determine which is best by after-tax analysis using an income tax rate of 40%, an after-tax MARR of 15%, and SL depreciation. Assume that each has a life of five years and no BV or MV. **(7.9)**

7-46. Allen International, Inc., manufactures chemicals. It needs to acquire a new piece of production equipment to work on production for a large order that Allen has received. The order is for a period of three years, and at the end of that time the machine would be sold.

Allen has received two supplier quotations, both of which will provide the required service. Quotation I has a first cost of $180,000 and an estimated salvage value of $50,000 at the end of three years. Its cost for operation and maintenance is estimated at $28,000 per year. Quotation II has a first cost of $200,000 and an estimated salvage value of $60,000 at the end of three years. Its cost for operation and maintenance is estimated at $17,000 per year. The company pays income tax at a rate of 40% on ordinary income and 28% on depreciation recovery. The machine will be depreciated using MACRS-GDS (asset class 28.0). Allen uses an after-tax MARR of 12% for economic analysis, and it plans to accept whichever of these two quotations costs less. **(7.10)**

To perform an after-tax analysis to determine which of these machines should be acquired, you must

a. state the study period you are using.

b. show all numbers necessary to support your conclusions.

c. state what the company should do.

7-47. AMT, Inc., is considering the purchase of a digital camera for maintenance of design specifications by feeding digital pictures directly into an engineering workstation where computer-aided design files can be superimposed over the digital pictures. Differences between the two images can be noted, and corrections, as appropriate, can then be made by design engineers. **(7.12)**

a. You have been asked by management to determine the PW of the EVA of this equipment, assuming the following estimates: capital investment = $345,000; market value at end of year six = $120,000; annual revenues = $120,000; annual expenses = $8,000; equipment life = 6 years; effective income tax rate = 50%; and after-tax MARR = 10% per year. MACRS depreciation will be used with a five-year recovery period.

b. Compute the PW of the equipment's ATCFs. Is your answer in Part (a) the same as your answer in Part (b)?

7-48. Refer to Example 7-15. Show that the PW of the annual EVA amounts by the new machinery is the same as the PW of the ATCF amounts ($17,208) given in Table 7-6. **(7.10, 7.11)**

7-49. Rework Example 7-21 using the MACRS depreciation method (assume three-year property class) instead of the SL depreciation method. **(7.11)**

7-50. *Extended Learning Exercise*
You have the option to purchase or lease a five-axis horizontal machining center. Any revenues generated from the operation of the machine will be the same whether it is leased or purchased. Considering the information given, should you lease or purchase the machine? Conduct after-tax analyses of both options. The effective income tax rate is 40%, the evaluation period is five years, and the MARR is 10% per year. NOTES: (1) Under the Lease Option, maintenance costs are included in the annual leasing cost. (2) Leasing costs are paid at the beginning of each year and are tax deductible. (3) Depreciation deductions cannot be taken on leased equipment. (4) Deposits are not tax deductible, and refunds of deposits are not taxable; however, owing to the difference in timing between payment and refund, they must be considered in your analysis. **(7.10)**

Leasing Option
Annual leasing cost: $55,000
Deposit (paid at EOY zero, refunded at EOY five): $75,000

Purchasing Option
Purchase price: $350,000 *capital to be borrowed at $i = 8\%$, equal annual payments (Principal + Interest) for three years*
Depreciation: three year, MACRS
Annual maintenance cost: $20,000
Resale value at EOY five: $150,000

7-51. Suppose that you invest $200 per month (before taxes) for 30 years (360 payments) and the annual interest rate (APR) is 8%, compounded monthly. If your income tax bracket is 28%, what lump sum, after-tax distribution can be taken at the end of 30 years? **(7.7)**

7-52. A $5,000 balance in a tax-deferred savings plan will grow to $50,313.50 in 30 years at an 8% per year interest rate. What would be the future worth if the $5,000 had been subject to a 28% income tax rate? **(7.7)**

7-53. Determine the after-tax yield (i.e., IRR on the ATCF) obtained by an individual who purchases a $10,000, 10-year, 10% nominal interest rate bond. The following information is given: **(7.7)**

- Interest is paid semi-annually, and the bond was bought after the fifth payment had just been received by the previous owner.
- The purchase price for the bond was $9,000.
- All revenues (including capital gains) are taxed at an income rate of 28%.
- The bond is held to maturity.

7-54. A 529-state-approved Individual Retirement Account (IRA) permits parents to invest tax-free dollars into their children's college education fund (this money may only be used for educational expenses). Another popular plan, the Roth IRA, requires after-tax dollars to be invested in a savings fund that may (or may not) be used for paying future college expenses. Both plans are tax free when the money is eventually withdrawn to assist with college expenses. Clearly, the 529 IRA plan is a better way to save for college expenses than the Roth IRA. Quantify "better" when the marginal income tax rate is 28% and $10,000 each year is invested in a mutual fund earning 8% per year for 10 years. *Note:* The estimated cost of a college education 10 years from now is $110,000. **(7.7)**

7-55. A Roth IRA enables an individual to invest after-tax dollars during the accumulation phase of a retirement plan. The money is then income tax free when it is withdrawn during retirement. A tax-deductible IRA, on the other hand, provides an up-front tax deduction for the annual contribution, but it then requires income taxes to be paid on all future distributions. A basic assumption as to which plan is more beneficial concerns the current income tax rates versus their projected rates in the future.

To illustrate, suppose that $2,000 is available to invest at the end of each year for 30 years. The income tax rate now and into the foreseeable future is 28%, so $2,000(1 − 0.28) = $1,440 is invested annually into the Roth IRA. However, $2,000 per year can be invested into a tax-deductible IRA. Money invested under either plan will be deposited into a mutual fund earning 8% per year, and all accumulated money will be withdrawn as a lumpsum at the end of year 30. **(7.7)**

a. Which plan is better if future distributions of the traditional (tax-free) IRA are taxed at an income tax rate of 28%?

b. Which plan is better if the future income tax rate at retirement (end of year 30) is 30%?

7-56. Some experts believe that a new car can lose as much as 25%–30% of its value within a week after the car leaves the dealership! Visit a local automobile dealership (or Web site), and verify that this loss in value is true (or not true). Compare your findings with those of other members of your class. **(7.2)**

Spreadsheet Exercises

7-57. A bowling alley costs $500,000 and has a useful life of 10 years. Its estimated MV at the end of year 10 is $20,000. Create a spreadsheet that calculates the depreciation for years 1–10 using (i) the SL method, (ii) the 200% DB method, and (iii) the MACRS method (GDS class life = 10 years). For each method, compute the PW of the depreciation deductions (at EOY 0). The MARR is 10% per year. If a large PW is desirable, what do you conclude regarding which method is preferred? **(7.5)**

7-58. Create a spreadsheet to solve Problem 7-24. What would the MV of M1 have to be (at the end of year five) for the firm to select M1? **(7.9)**

7-59. Interest on municipal bonds is usually exempt from federal income taxes. The interest rate on these bonds is therefore an *after-tax* rate of return (ROR). Other types of bonds (e.g., corporate bonds) pay interest that is taxable for federal income tax purposes. Thus, the before-tax ROR on such bonds is typically higher than the ROR on municipal bonds. Develop a spreadsheet that contains the before-tax ROR (on taxable bonds) that are equivalent to after-tax RORs of 4%, 5%, and 6% for income tax rates of 15%, 28%, and 35%. (7.6)

7-60. Refer to the chapter opener and Example 7-14. As an alternative to the coal-fired plant, PennCo could construct an 800 MW natural gas-fired plant. This plant would require the same initial investment of \$1.12 billion dollars to be depreciated over its 30-year life using the SL method with $SV_{30} = 0$. The capacity factor estimate of the plant would still be 80%, but the efficiency of the natural gas-fired plant would be 40%. The annual operating and maintenance expense is expected to be \$0.01 per kWh. The cost of natural gas is \$8.00 per million Btu and the carbon dioxide tax is \$15 per metric ton. Natural gas emits 55 metric tons of carbon dioxide per billion Btu produced. The effective income tax rate is 40%, and the after-tax MARR is 10% per year. Based on the after-tax cost of electricity, create a spreadsheet to determine whether PennCo should construct a natural gas-fired or coal-fired plant. *Note:* 1 kWh = 3,413 Btu. (7.9)

FE Practice Problems

The Parkview Hospital is considering the purchase of a new autoclave. This equipment will cost \$250,000. This asset will be depreciated using an MACRS (GDS) recovery period of three years. Use this information to solve Problems 7-61 to 7-63.

7-61. The depreciation amount in the first year is

(a) \$50,000 (b) \$66,675 (c) \$83,325
(d) \$55,563

7-62. The BV at the end of the third year is

(a) \$27,771 (b) \$18,525 (c) \$116,675
(d) \$33,325

7-63. If the autoclave is sold during the third year of ownership, the allowable depreciation charge for the third year is

(a) \$25,000 (b) \$37,025 (c) \$22,215
(d) \$11,108

An oil refinery has decided to purchase some new drilling equipment for \$550,000. The equipment will be kept for 10 years before being sold. The estimated SV for depreciation purposes is to be \$25,000. Use this information to solve Problems 7-64 to 7-67.

7-64. Using the SL method, the annual depreciation on the equipment is

(a) \$50,000 (b) \$51,500 (c) \$52,500
(d) \$55,000

7-65. Using the SL method, the BV at the end of the depreciable life is

(a) \$0 (b) \$25,000 (c) \$35,000
(d) \$50,000

7-66. If SL depreciation is used and the equipment is sold for \$35,000 at the end of 10 years, the taxable gain on the disposal of the equipment is

(a) \$35,000 (b) \$25,000 (c) \$15,000
(d) \$10,000

7-67. If MACRS depreciation is used, the recovery period of the equipment using the GDS guidelines is

(a) 3 years (b) 5 years (c) 7 years
(d) 10 years

A wood products company has decided to purchase new logging equipment for \$100,000 with a trade-in of its old equipment. The old equipment has a BV of \$10,000 at the time of the trade-in. The new equipment will be kept for 10 years before being sold. Its estimated SV at the time is expected to be \$5,000. Use this information to solve Problems 7-68 through 7-72.

7-68. The recovery period of the asset, using the GDS guidelines, is

(a) 10 years (b) 7 years (c) 5 years
(d) 3 years

7-69. Using the SL method, the depreciation on the equipment over its depreciable life period is

(a) \$10,500 (b) \$9,500 (c) \$8,000
(d) \$7,000

7-70. Using the SL method, the BV at the end of the depreciable life is

(a) \$11,811 (b) \$10,000 (c) \$5,000 (d) \$0

7-71. Using the MACRS (GDS recovery period), the depreciation charge permissible at year 6 is equal to

(a) \$9,812 (b) \$6,336 (c) \$4,912 (d) \$0

7-72. Using the MACRS (GDS recovery period), if the equipment is sold in year five, the BV at the end of year five is equal to

(a) \$29,453 (b) \$24,541 (c) \$12,672
(d) \$6,336

7-73. A small pump costs \$20,000 and has a life of ten years and a \$2,000 SV at that time. If the 200% DB method is used to depreciate the pump, the BV at the end of year four is

(a) \$9,000 (b) \$8,192 (c) \$6,000
(d) \$5,000

7-74. Air handling equipment that costs \$14,000 has a life of six years with a \$2,000 SV. What is the SL depreciation amount for each year?

(a) \$1,500 (b) \$1,000 (c) \$1,200
(d) \$2,000

7-75. The air handling equipment just described is to be depreciated, using the MACRS with a GDS recovery period of five years. The BV of the equipment at the end of (including) year four is most nearly

(a) \$3,749 (b) \$3,124 (c) \$24,192
(d) \$8,251

7-76. If the federal income tax rate is 45% and the state tax rate is 15% (and state taxes are deductible from federal taxes), the effective income tax rate is

(a) 35% (b) 37.5% (c) 38.3%
(d) 53.3%

7-77. If a company's total effective income tax rate is 40% and its state income tax rate is 20%, what is the company's federal income tax rate?

(a) 20% (b) 25% (c) 35%
(d) 40% (e) 52%

7-78. Acme Manufacturing makes their preliminary economic studies using a before-tax MARR of 18%. More detailed studies are performed on an after-tax basis. If their effective tax rate is 40%, the after-tax MARR is

(a) 6% (b) 7% (c) 11% (d) 13%

7-79. Suppose for some year the income of a small company is \$110,000; the expenses are \$65,000; the depreciation is \$25,000; and the effective income tax rate = 40%. For this year, the ATCF is most nearly

(a) −\$8,900 (b) \$4,700 (c) \$13,200
(d) \$29,700 (e) \$37,000

Your company is contemplating the purchase of a large stamping machine. The machine will cost \$180,000. With additional transportation and installation costs of \$5,000 and \$10,000, respectively, the cost basis for depreciation purposes is \$195,000. Its MV at the end of five years is estimated as \$40,000. The IRS has assured you that this machine will fall under a three-year MACRS class life category. The justifications for this machine include \$40,000 savings per year in labor and \$30,000 savings per year in reduced materials. The before-tax MARR is 20% per year, and the effective income tax rate is 40%. Use this information to solve problems 7-80 through 7-83.

7-80. The *total* before-tax cash flow in year five is most nearly (assuming you sell the machine at the end of year five):

(a) \$9,000 (b) \$40,000 (c) \$70,000
(d) \$80,000 (e) \$110,000

7-81. The taxable income for year three is most nearly

(a) \$5,010 (b) \$16,450 (c) \$28,880
(d) \$41,120 (e) \$70,000

7-82. The PW of the *after-tax savings* from the machine, *in labor and materials only*, (neglecting the first cost, depreciation, and the salvage value) is most nearly (using the after tax MARR)

(a) \$12,000 (b) \$95,000 (c) \$151,000
(d) \$184,000 (e) \$193,000

7-83. Assume the stamping machine will now be used for only three years, owing to the company's losing several government contracts. The MV at the end of year three is \$50,000. What is the income tax owed at the end of year three owing to depreciation recapture (capital gain)?

(a) \$8,444 (b) \$14,220 (c) \$21,111
(d) \$35,550 (e) \$20,000

7-84. Given a MARR of 10%, which alternative should the company select? (7.10)

Alternative	IRR	PW(10%)
A	18.2%	\$12,105
B	15.6%	\$12,432

(a) A (b) B (c) Do Nothing

ase Study Exercises

29. Determine how much the annual lease amount of e challenger can increase before the defender becomes e preferred alternative. (9.10)

30. Suppose the study period is reduced to five years. Vill this change the replacement decision? (9.10)

31. Before committing to a 10-year lease, it was ecided to consider one more alternative. Instead of simply augmenting the existing generator with a new 40-kW unit, this alternative calls for replacing the 80-kW generator with two 40-kW units (for a total of three 40-kW units). The company selling the 40-kW units will reduce the purchase price to $120,000 per generator if three generators are bought. How does this new challenger compare to the augmented defender and the original challenger? (9.10)

E Practice Problems

-32. Machine A was purchased last year for $18,000 nd had an estimated MV of $2,000 at the end of its ix-year life. Annual operating costs are $1,600. The machine will perform satisfactorily over the next five years. A salesman for another company is offering a replacement, Machine B, for $14,000, with an MV of $1,500 after five years. Annual operating costs for Machine B will only be $1,200. A trade-in allowance of $8,400 has been offered for Machine A. If the before-tax MARR is 12% per year, should you buy the new machine? (9.4)

(a) No, continue with Machine A.
(b) Yes, purchase Machine B.

9-33. A company is considering replacing a machine that was bought six years ago for $60,000. The machine, however, can be repaired and its life extended by five more years. If the current machine is replaced, the new machine will cost $45,000 and will reduce the operating expenses by $8,000 per year. The seller of the new machine has offered a trade-in allowance of $12,000 for the old machine. If MARR is 12% per year before taxes, how much can the company spend to repair the existing machine? Choose the closest answer. (9.4)

(a) $22,371 (b) $50,628 (c) $4,161 (d) −$1,000

9-34. Using the outsider viewpoint, what is the EUAC of continuing to use Machine A? Choose the closest answer. (9.6)

(a) $1,000 (b) $2,182 (c) $2,713
(d) $901 (e) $2,435

9-35. Using the outsider viewpoint, what is the EUAC of buying Machine B? Choose the closest answer. (9.5)

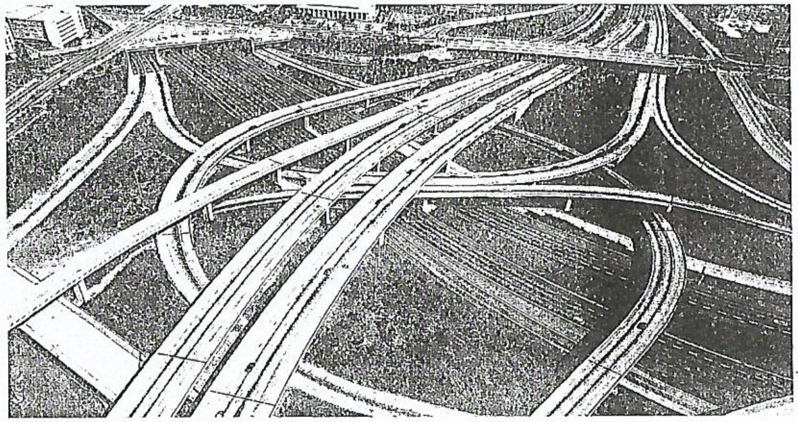
(a) $8,565 (b) $11,361 (c) $9,750
(d) $9,165 (e) $900

9-36. A corporation purchased a machine for $60,000 five years ago. It had an estimated life of 10 years and an estimated salvage value of $9,000. The current BV of this machine is $34,500. If the current MV of the machine is $40,500 and the effective income tax rate is 29%, what is the after-tax investment value of the machine? Use the outsider viewpoint. (9.9)

(a) $28,755 (b) $40,500 (c) $38,760
(d) $37,455 (e) $36,759

CHAPTER 10

Evaluating Projects with the Benefit–Cost Ratio Method



The objective of Chapter 10 is to demonstrate the use of the benefit–cost (B–C) ratio for the evaluation of public projects.

Constructing a Bypass to Relieve Traffic Congestion

Two heavily traveled interstate highways currently intersect in the middle of a major metropolitan area. A 25-mile, four-lane bypass is being considered to connect the busy interstate highways at a point outside of the metropolitan area. The projected construction cost of the bypass is $20 million. Annual maintenance of the roadway is expected to be $500,000. Major monetary benefits of reduced time delays due to traffic congestion, improved traveler safety (fewer traffic accidents), and expanded opportunities for commercial businesses are anticipated to be around $2 million per year.

This bypass would be state owned and maintained and is therefore considered a public project. In this chapter, you will learn how public projects such as this are evaluated using a benefit–cost ratio.

> ... the Federal Government should improve or participate in the improvement of navigable waters or their tributaries, including watersheds thereof, for flood-control purposes if the benefits to whomsoever they may accrue are in excess of the estimated costs ...
>
> —Flood Control Act (1936)

10.1 Introduction

Public projects are those authorized, financed, and operated by federal, state, or local governmental agencies. Such public works are numerous, and although they may be of any size, they are frequently much larger than private ventures. Since they require the expenditure of capital, such projects are subject to the principles of engineering economy with respect to their design, acquisition, and operation. Because they are public projects, however, a number of important special factors exist that are not ordinarily found in privately financed and operated businesses. The differences between public and private projects are listed in Table 10-1.

TABLE 10-1 Some Basic Differences between Privately Owned and Publicly Owned Projects

	Private	Public
Purpose	Provide goods or services at a profit; maximize profit or minimize cost	Protect health; protect lives and property; provide services (at no profit); provide jobs
Sources of capital	Private investors and lenders	Taxation; private lenders
Method of financing	Individual ownership; partnerships; corporations	Direct payment of taxes; loans without interest; loans at low interest; self-liquidating bonds; indirect subsidies; guarantee of private loans
Multiple purposes	Moderate	Common (e.g., reservoir project for flood control, electrical power generation, irrigation, recreation, education)
Project life	Usually relatively short (5 to 10 years)	Usually relatively long (20 to 60 years)
Relationship of suppliers of capital to project	Direct	Indirect, or none
Nature of benefits	Monetary or relatively easy to equate to monetary terms	Often nonmonetary, difficult to quantify, difficult to equate to monetary terms
Beneficiaries of project	Primarily, entity undertaking project	General public
Conflict of purposes	Moderate	Quite common (dam for flood control versus environmental preservation)
Conflict of interests	Moderate	Very common (between agencies)
Effect of politics	Little to moderate	Frequent factors; short-term tenure for decision makers; pressure groups; financial and residential restrictions; etc.
Measurement of efficiency	Rate of return on capital	Very difficult; no direct comparison with private projects

As a consequence of these differences, it is often difficult to make engineering economy studies and investment decisions for public-works projects in exactly the same manner as for privately owned projects. Different decision criteria are often used, which creates problems for the public (who pays the bill), for those who must make the decisions, and for those who must manage public-works projects.

The benefit–cost ratio method, which is normally used for the evaluation of public projects, has its roots in federal legislation. Specifically, the Flood Control Act of 1936 requires that, for a federally financed project to be justified, its benefits must be in excess of its costs. In general terms, B–C analysis is a systematic method of assessing the desirability of government projects or policies when it is important to take a long-term view of future effects and a broad view of possible side effects. In meeting the requirements of this mandate, the B–C ratio method evolved into the calculation of a ratio of project benefits to project costs. Rather than allowing the analyst to apply criteria more commonly used for evaluating private projects (IRR, PW, and so on), most governmental agencies require the use of the B–C method.

10.2 Perspective and Terminology for Analyzing Public Projects

Before applying the B–C ratio method to evaluate a public project, the appropriate perspective (Chapter 1, Principle 3) must be established. In conducting an engineering economic analysis of any project, whether it is a public or a private undertaking, the proper perspective is to consider the *net* benefits to the *owners* of the enterprise considering the project. This process requires that the question of who owns the project be addressed. Consider, for example, a project involving the expansion of a section of I-80 from four to six lanes. Because the project is paid for primarily with federal funds channeled through the Department of Transportation, we might be inclined to say that the federal government is the owner. These funds, however, originated from tax dollars—thus, the true owners of the project are the taxpayers.

As mentioned previously, the B–C method requires that a ratio of benefits to costs be calculated. Project *benefits* are defined as the favorable consequences of the project to the public, but project *costs* represent the monetary disbursement(s) required of the government. It is entirely possible, however, for a project to have unfavorable consequences to the public. Considering again the widening of I-80, some of the *owners* of the project—farmers along the interstate—would lose a portion of their arable land, along with a portion of their annual revenues. Because this negative financial consequence is borne by (a segment of) the public, it cannot be classified as either a benefit or a cost. The term *disbenefits* is generally used to represent the negative consequences of a project to the public.

EXAMPLE 10-1 Benefits and Costs of a Convention Center and Sports Complex

A new convention center and sports complex has been proposed to the Gotham City Council. This public-sector project, if approved, will be financed through the issue of municipal bonds. The facility will be located in the City Park near downtown Gotham City, in a wooded area, which includes a bike path, a nature trail, and a pond. Because the city already owns the park, no purchase of land is necessary. List separately the project's *benefits, costs,* and any *disbenefits.*

Solution

BENEFITS	Improvement of the image of the downtown area of Gotham City
	Potential to attract conferences and conventions to Gotham City
	Potential to attract professional sports franchises to Gotham City
	Revenues from rental of the facility
	Increased revenues for downtown merchants of Gotham City
	Use of facility for civic events
COSTS	Architectural design of the facility
	Construction of the facility
	Design and construction of parking garage adjacent to the facility
	Operating and maintenance costs of the facility
	Insurance costs of the facility
DISBENEFITS	Loss of use of a portion of the City Park to Gotham City residents, including the bike path, the nature trail, and the pond
	Loss of wildlife habitat in urban area

10.3 Self-Liquidating Projects

The term *self-liquidating project* is applied to a governmental project that is expected to earn direct revenue sufficient to repay its cost in a specified period of time. Most of these projects provide utility services—for example, the fresh water, electric power, irrigation water, and sewage disposal provided by a hydroelectric dam. Other examples of self-liquidating projects include toll bridges and highways.

As a rule, self-liquidating projects are expected to earn direct revenues that offset their costs, but they are not expected to earn profits or pay income taxes. Although they also do not pay property taxes, in some cases in-lieu payments are made to state, county, or municipal governments in place of the property or franchise taxes that would have been paid had the project been under private ownership. For example, the U.S. government agreed to pay the states of Arizona and Nevada $300,000 each annually for 50 years in lieu of taxes that would have accrued if Hoover Dam had been privately constructed and operated.

10.4 Multiple-Purpose Projects

An important characteristic of public-sector projects is that many such projects have multiple purposes or objectives. One example of this would be the construction of a dam to create a reservoir on a river. (See Figure 10-1.) This project would have multiple purposes: (1) assist in flood control, (2) provide water for irrigation,

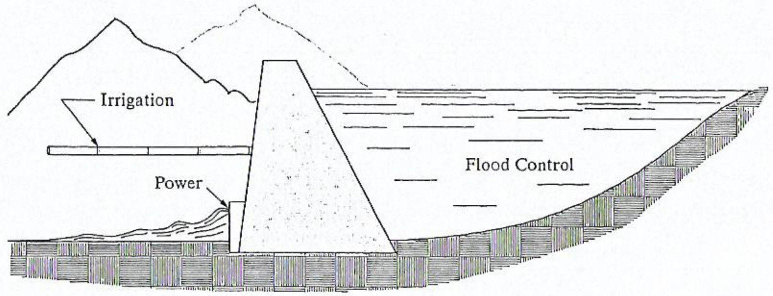


Figure 10-1 Schematic Representation of a Multiple-Purpose Project Involving Flood Control, Irrigation, and Power

(3) generate electric power, (4) provide recreational facilities, and (5) provide drinking water. Developing such a project to meet more than one objective ensures that greater overall economy can be achieved. Because the construction of a dam involves very large sums of capital and the use of a valuable natural resource—a river—it is likely that the project could not be justified unless it served multiple purposes. This type of situation is generally desirable, but, at the same time, it creates economic and managerial problems due to the overlapping utilization of facilities and the possibility of a conflict of interest between the several purposes and the agencies involved.

The basic problems that often arise in evaluating public projects can be illustrated by returning to the dam shown in Figure 10-1. The project under consideration is to be built in the semiarid central portion of California, primarily to provide control against spring flooding resulting from the melting snow in the Sierra Nevadas. If a portion of the water impounded behind the dam could be diverted onto the adjoining land below the dam, the irrigation water would greatly increase the productivity, and thus the value, of that land, which would result in an increase in the nation's resources. So the objectives of the project should be expanded to include both flood control and irrigation.

The existence of a dam with a high water level on one side and a much lower level on the other side also suggests that some of the nation's resources will be wasted unless a portion of the water is diverted to run through turbines, generating electric power. This electricity can be sold to customers in the areas surrounding the reservoir, giving the project the third purpose of generating electric power.

In this semiarid region, the creation of a large reservoir behind the dam would provide valuable facilities for hunting, fishing, boating, swimming, and camping. Thus, the project has the fourth purpose of providing recreation facilities. A fifth purpose would be the provision of a steady, reliable supply of drinking water.

Each of the above-mentioned objectives of the project has desirable economic and social value, so what started out as a single-purpose project now has five purposes. The failure to meet all five objectives would mean that valuable national resources are being wasted. On the other hand, there are certain *disbenefits* to the public that must also be considered. Most apparent of these is the loss of farm land above the dam in the area covered by the reservoir. Other disbenefits might include (1) the loss of a white-water recreational area enjoyed by canoeing, kayaking, and rafting enthusiasts, (2) the loss of annual deposits of fertile soil in the river basin below the dam due to spring flooding, and (3) the negative ecological impact of obstructing the flow of the river.

If the project is built to serve five purposes, the fact that one dam will serve all of them leads to at least three basic problems. The *first* of these is the allocation of the cost of the dam to each of its intended purposes. Suppose, for example, that the estimated costs of the project are $35,000,000. This figure includes costs incurred for the purchase and preparation of land to be covered by water above the dam site; the actual construction of the dam, the irrigation system, the power generation plant, and the purification and pumping stations for drinking water; and the design and development of recreational facilities. The allocation of some of these costs to specific purposes is obvious (e.g., the cost of constructing the irrigation system), but what portion of the cost of purchasing and preparing the land should be assigned to flood control? What amounts should be assigned to irrigation, power generation, drinking water, and recreation?

The *second* basic problem is the conflict of interest among the several purposes of the project. Consider the decision as to the water level to be maintained behind the dam. In meeting the first purpose—flood control—the reservoir should be maintained at a near-empty level to provide the greatest storage capacity during the months of the spring thaw. This lower level would be in direct conflict with the purpose of power generation, which could be maximized by maintaining as high a level as possible behind the dam at all times. Further, maximizing the recreational benefits would suggest that a constant water level be maintained throughout the year. Thus, conflicts of interest arise among the multiple purposes, and compromising decisions must be made. These decisions ultimately affect the magnitude of benefits resulting from the project.

A *third* problem with multiple-purpose public projects is political sensitivity. Because each of the various purposes, or even the project itself, is likely to be desired or opposed by some segment of the public and by various interest groups that may be affected, inevitably such projects frequently become political issues.* This conflict often has an effect on cost allocations and thus on the overall economy of these projects.

The net result of these three factors is that the cost allocations made in multiple-purpose public-sector projects tend to be arbitrary. As a consequence, production and selling costs of the services provided also are arbitrary. Because of this, these costs cannot be used as valid yardsticks with which similar private-sector projects can be compared to determine the relative efficiencies of public and private ownership.

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* The construction of the Tellico Dam on the Little Tennessee River was considerably delayed due to two important *disbenefits:* (1) concerns over the impact of the project on the environment of a small fish, the snail darter, and (2) the flooding of burial grounds considered sacred by the Cherokee Nation.

10.5 Difficulties in Evaluating Public-Sector Projects

With all of the difficulties that have been cited in evaluating public-sector projects, we may wonder whether engineering economy studies of such projects should be attempted. In most cases, economy studies cannot be made in as complete, comprehensive, and satisfactory a manner as in the case of studies of privately financed projects. In the private sector, the *costs* are borne by the firm undertaking the project, and the *benefits* are the favorable outcomes of the project accrued by the firm. Any costs and benefits that occur outside of the firm are generally ignored in evaluations unless it is anticipated that those external factors will indirectly impact the firm. But the opposite is true in the case of public-sector projects. In the wording of the Flood Control Act of 1936, "if the benefits to whomsoever they may accrue are in excess of the estimated costs," all of the potential benefits of a public project are relevant and should be considered. Simply enumerating all of the benefits for a large-scale public project is a formidable task! Further, the monetary value of these benefits to all of the affected segments of the public must somehow be estimated. Regardless, decisions about the investment of capital in public projects must be made by elected or appointed officials, by managers, or by the general public in the form of referendums. Because of the magnitude of capital and the long-term consequences associated with many of these projects, following a systematic approach for evaluating their worthiness is vital.

There are a number of difficulties inherent in public projects that must be considered in conducting engineering economy studies and making economic decisions regarding those projects. Some of these are as follows:

1. There is no profit standard to be used as a measure of financial effectiveness. Most public projects are intended to be nonprofit.
2. The monetary impact of many of the benefits of public projects is difficult to quantify.
3. There may be little or no connection between the project and the public, which is the owner of the project.
4. There is often strong political influence whenever public funds are used. When decisions regarding public projects are made by elected officials who will soon be seeking reelection, *the immediate benefits and costs are stressed, often with little or no consideration for the more important long-term consequences.*
5. Public projects are usually much more subject to legal restrictions than are private projects. For example, the area of operations for a municipally owned power company may be restricted such that power can be sold only within the city limits, regardless of whether a market for any excess capacity exists outside the city.
6. The ability of governmental bodies to obtain capital is much more restricted than that of private enterprises.
7. The appropriate interest rate for discounting the benefits and costs of public projects is often controversially and politically sensitive. Clearly, lower interest rates favor long-term projects having major social or monetary benefits in the future. Higher interest rates promote a short-term outlook whereby decisions are based mostly on initial investments and immediate benefits.

A discussion of several viewpoints and considerations that are often used to establish an appropriate interest rate for public projects is included in the next section.

10.6 What Interest Rate Should Be Used for Public Projects?

When public-sector projects are evaluated, interest rates* play the same role of accounting for the time value of money as in the evaluation of projects in the private sector. The rationale for the use of interest rates, however, is somewhat different. The choice of an interest rate in the private sector is intended to lead directly to a selection of projects to maximize profit or minimize cost. In the public sector, on the other hand, projects are not usually intended as profit-making ventures. Instead, the goal is the *maximization of social benefits*, assuming that these have been appropriately measured. The choice of an interest rate in the public sector is intended to determine how available funds should best be allocated among competing projects to achieve social goals. The relative differences in magnitude of interest rates between governmental agencies, regulated monopolies, and private enterprises are illustrated in Figure 10-2.

Three main considerations bear on what interest rate to use in engineering economy studies of public-sector projects:

1. The interest rate on borrowed capital
2. The opportunity cost of capital to the governmental agency
3. The opportunity cost of capital to the taxpayers

As a general rule, it is appropriate to use the interest rate on borrowed capital as the interest rate for cases in which money is borrowed specifically for the project(s) under consideration. For example, if municipal bonds are issued specifically for the financing of a new school, the effective interest rate on those bonds should be the interest rate.

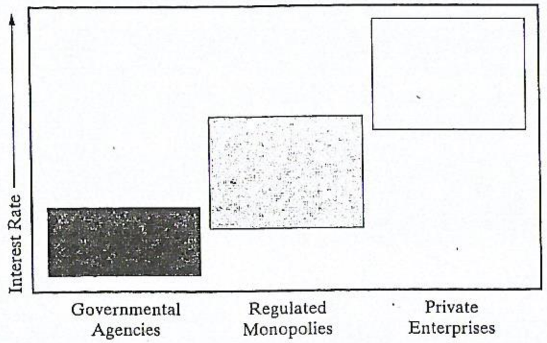


Figure 10-2 Relative Differences in Interest Rates for Governmental Agencies, Regulated Monopolies, and Private Enterprises

* The term *discount rate* is often used instead of interest rate (or MARR) in government literature on B–C analysis.

For public-sector projects, the opportunity cost of capital to a governmental agency encompasses the annual rate of *benefit* to either the constituency served by that agency or the composite of taxpayers who will eventually pay for the project. If projects are selected such that the estimated *return* (in terms of benefits) on all accepted projects is higher than that on any of the rejected projects, then the interest rate used in economic analyses is that associated with the best opportunity forgone. If this process is done for all projects and investment capital available within a governmental agency, the result is *an opportunity cost of capital for that governmental agency.* A strong argument against this philosophy, however, is that the different funding levels of the various agencies and the different nature of projects under the direction of each agency would result in different interest rates for each of the agencies, even though they all share a common primary source of funds—taxation of the public.

The third consideration—the opportunity cost of capital to the taxpayers—is based on the philosophy that all government spending takes potential investment capital away from the taxpayers. The taxpayers' opportunity cost is generally greater than either the cost of borrowed capital or the opportunity cost to governmental agencies, and there is a compelling argument for applying the largest of these three rates as the interest rate for evaluating public projects; it is not economically sound to take money away from a taxpayer to invest in a government project yielding benefits at a rate less than what could have been earned by that taxpayer.

This argument was supported by a federal government directive issued in 1997—and still in force—by the Office of Management and Budget (OMB).* According to this directive, a 7% interest rate should be used in economic evaluations for a wide range of federal projects, with certain exceptions (e.g., a lower rate can be applied in evaluating water resource projects). This 7%, it can be argued, is at least a rough approximation of the real-dollar return that taxpayers could earn from the use of that money for private investment. This corresponds to an approximate nominal interest rate of 10% per year.

One additional theory on establishing interest rates for federal projects advocates that the *social discount rate* used in such analyses should be the market-determined *risk-free* rate for private investments.† According to this theory, a nominal interest rate in the order of 3–4% per year should be used.

The preceding discussion focuses on the *considerations* that should play a role in establishing an interest rate for public projects. As in the case of the private sector, there is no simple formula for determining the appropriate interest rate for public projects. With the exception of projects falling under the 1997 OMB directive, setting the interest rate is ultimately a policy decision at the discretion of the governmental agency conducting the analysis.

* Office of Management and Budget, "Guidelines and Discount Rates for Benefit–Costs Analysis of Federal Programs," *OMB Circular No. A-94 (revised)*, February 21, 1997. The OMB home page is `http://www.whitehouse.gov/omb`.
† K. J. Arrow and R. C. Lind, "Uncertainty and the Evaluation of Public Investment Decisions," *American Economic Review*, 60 (June 1970): 364–378.

10.7 The Benefit–Cost Ratio Method

As the name implies, the B–C ratio method involves the calculation of a ratio of benefits to costs. Whether evaluating a project in the private sector or in the public sector, the time value of money must be considered to account for the timing of cash flows (or benefits) occurring after the inception of the project. Thus, the B–C ratio is actually a ratio of *discounted benefits* to *discounted costs*.

Any method for formally evaluating projects in the public sector must consider the worthiness of allocating resources to achieve social goals. For over 70 years, the B–C ratio method has been the accepted procedure for making go/no-go decisions on independent projects and for comparing mutually exclusive projects in the public sector, even though the other methods discussed in Chapter 5 (PW, AW, IRR, etc.) will lead to identical recommendations, *assuming all these procedures are properly applied.*

Accordingly, the purpose of this section is to describe and illustrate the mechanics of the B–C ratio method for evaluating projects. Two different B–C ratios will be presented because they are used in practice by various government agencies and municipalities. Both ratios lead to the *identical choice* of which project is best when comparing mutually exclusive alternatives.

The B–C ratio is defined as the ratio of the equivalent worth of benefits to the equivalent worth of costs. The equivalent-worth measure applied can be present worth, annual worth, or future worth, but customarily, either PW or AW is used. An interest rate for public projects, as discussed in the previous section, is used in the equivalent-worth calculations. The B–C ratio is also known as the *savings-investment ratio* (*SIR*) by some governmental agencies.

Several different formulations of the B–C ratio have been developed. Two of the more commonly used formulations are presented in this section, illustrating the use of both present worth and annual worth.

Conventional B–C ratio with PW:

$$\text{B–C} = \frac{\text{PW(benefits of the proposed project)}}{\text{PW(total costs of the proposed project)}} = \frac{\text{PW}(B)}{I - \text{PW(MV)} + \text{PW(O\&M)}} \qquad (10\text{-}1)$$

where PW(·) = present worth of (·);
B = benefits of the proposed project;
I = initial investment in the proposed project;
MV = market value at the end of useful life;
O&M = operating and maintenance costs of the proposed project.

Modified B–C ratio with PW:

$$\text{B–C} = \frac{\text{PW}(B) - \text{PW(O\&M)}}{I - \text{PW(MV)}}. \qquad (10\text{-}2)$$

The numerator of the modified B–C ratio expresses the equivalent worth of the benefits minus the equivalent worth of the operating and maintenance costs, and the denominator includes only the initial investment costs (less any market value). A project is acceptable when the B–C ratio, as defined in either Equation (10-1) or (10-2), is greater than or equal to 1.0.

Equations (10-1) and (10-2) can be rewritten in terms of equivalent annual worth, as follows:

Conventional B–C ratio with AW:

$$\text{B–C} = \frac{\text{AW(benefits of the proposed project)}}{\text{AW(total costs of the proposed project)}} = \frac{\text{AW}(B)}{\text{CR} + \text{AW(O\&M)}}, \qquad (10\text{-}3)$$

where AW(·) = annual worth of (·);
B = benefits of the proposed project;
CR = capital-recovery amount (i.e., the equivalent annual cost of the initial investment, I, including an allowance for market, or salvage value, if any);
O&M = operating and maintenance costs of the proposed project.

Modified B–C ratio with AW:

$$\text{B–C} = \frac{\text{AW}(B) - \text{AW(O\&M)}}{\text{CR}}. \qquad (10\text{-}4)$$

The resulting B–C ratios for all the previous formulations will give identical results in determining the acceptability of a project (i.e., either B–C $\geq$ 1.0 or B–C $<$ 1.0). The conventional B–C ratio will give identical numerical results for both PW and AW formulations; similarly, the modified B–C ratio gives identical numerical results whether PW or AW is used. Although the magnitude of the B–C ratio will differ between conventional and modified B–C ratios, go/no-go decisions are not affected by the choice of approach, as shown in Example 10-2.

EXAMPLE 10-2 **Equivalence of the B–C Ratio Formulations**

The city of Columbia is considering extending the runways of its municipal airport so that commercial jets can use the facility. The land necessary for the runway extension is currently a farmland that can be purchased for \$350,000. Construction costs for the runway extension are projected to be \$600,000, and the additional annual maintenance costs for the extension are estimated to be \$22,500. If the runways are extended, a small terminal will be constructed at a cost of \$250,000. The annual operating and maintenance costs for the terminal are estimated at \$75,000. Finally, the projected increase in flights will require the addition of two air traffic controllers at an annual cost of \$100,000. Annual *benefits* of the runway extension have been estimated as follows:

\$325,000	Rental receipts from airlines leasing space at the facility
\$65,000	Airport tax charged to passengers
\$50,000	Convenience benefit for residents of Columbia
\$50,000	Additional tourism dollars for Columbia

Apply the B–C ratio method with a study period of 20 years and a MARR of 10% per year to determine whether the runways at Columbia Municipal Airport should be extended.

Solution

Conventional B–C: Equation (10-1)	$\text{B–C} = \text{PW}(B)/[I - \text{PW(MV)} + \text{PW(O\&M)}]$ $\text{B–C} = \$490{,}000\,(P/A, 10\%, 20)/[\$1{,}200{,}000 + \$197{,}500\,(P/A, 10\%, 20)]$ **B–C = 1.448 > 1; extend runways.**
Modified B–C: Equation (10-2)	$\text{B–C} = [\text{PW}(B) - \text{PW(O\&M)}]/[I - \text{PW(MV)}]$ $\text{B–C} = [\$490{,}000\,(P/A, 10\%, 20) - \$197{,}500\,(P/A, 10\%, 20)]/\$1{,}200{,}000$ **B–C = 2.075 > 1; extend runways.**
Conventional B–C: Equation (10-3)	$\text{B–C} = \text{AW}(B)/[\text{CR} + \text{AW(O\&M)}]$ $\text{B–C} = \$490{,}000/[\$1{,}200{,}000\,(A/P, 10\%, 20) + \$197{,}500]$ **B–C = 1.448 > 1; extend runways.**
Modified B–C: Equation (10-4)	$\text{B–C} = [\text{AW}(B) - \text{AW(O\&M)}]/\text{CR}$ $\text{B–C} = [\$490{,}000 - \$197{,}500]/[\$1{,}200{,}000\,(A/P, 10\%, 20)]$ **B–C = 2.075 > 1; extend runways.**

As can be seen in the preceding example, the difference between conventional and modified B-C ratios is essentially due to subtracting the equivalent-worth measure of operating and maintenance costs from both the numerator and the denominator of the B-C ratio. In order for the B-C ratio to be greater than 1.0, the numerator must be greater than the denominator. Similarly, the numerator must be less than the denominator for the B-C ratio to be less than 1.0. Subtracting a constant (the equivalent worth of operating and maintenance costs) from both numerator and denominator does not alter the *relative* magnitudes of the numerator and denominator. Thus, project acceptability is not affected by the choice of conventional versus modified B-C ratio. This information is stated mathematically as follows for the case of B-C > 1.0:

Let N = the numerator of the conventional B–C ratio,
D = the denominator of the conventional B–C ratio,
O&M = the equivalent worth of operating and maintenance costs.

$$\text{If B–C} = \frac{N}{D} > 1.0, \text{ then } N > D.$$

$$\text{If } N > D, \text{ and } [N - \text{O\&M}] > [D - \text{O\&M}], \text{ then } \frac{N - \text{O\&M}}{D - \text{O\&M}} > 1.0.$$

Note that $\frac{N - \text{O\&M}}{D - \text{O\&M}}$ *is the modified* B–C *ratio; thus, if conventional* B–C > 1.0, *then modified* B–C > 1.0

EXAMPLE 10-3 **B–C Ratio of the Proposed Bypass**

In this example, we will evaluate the bypass described in the beginning of the chapter. The construction cost of the bypass is \$20 million, and \$500,000 would be required each year for annual maintenance. The annual benefits to the public have been estimated to be \$2 million. If the study period is 50 years and the state's interest rate is 8% per year, should the bypass be constructed? What impact does a social interest rate of 4% per year have on the B-C ratio of the project?

Solution

At an interest rate of 8% per year, the conventional B–C ratio of the proposed bypass is

$$\text{B–C} = \frac{\$2{,}000{,}000}{\$20{,}000{,}000\,(A/P,\ 8\%,\ 50) + \$500{,}000} = 0.94.$$

Because this ratio is less than one, the bypass is not economically acceptable at 8% interest.

If a social interest rate of 4% per year was used, the B–C ratio would be 1.40 and the bypass would be acceptable.

Two additional issues of concern are the treatment of *disbenefits* in B–C analyses and the decision as to whether certain cash flow items should be treated as *additional benefits* or as *reduced costs*. The first concern arises whenever disbenefits are formally defined in a B–C evaluation of a public-sector project. An example of the second concern would be a public-sector project proposing to replace an existing asset having high annual operating and maintenance costs with a new asset having lower operating and manual costs. As will be seen in Sections 10.7.1 and 10.7.2, the final recommendation on a project is not altered by either the approach to incorporating disbenefits or the classification of an item as a reduced cost or an additional benefit.

10.7.1 Disbenefits in the B–C Ratio

In a previous section, disbenefits were defined as negative consequences to the public resulting from the implementation of a public-sector project. The traditional approach for incorporating disbenefits into a B–C analysis is to reduce benefits by the amount of disbenefits (i.e., to subtract disbenefits from benefits in the numerator of the B–C ratio). Alternatively, the disbenefits could be treated as costs (i.e., add disbenefits to costs in the denominator). Equations (10-5) and (10-6) illustrate the two approaches for incorporating disbenefits in the conventional B–C ratio, with benefits, costs, and disbenefits in terms of equivalent AW. (Similar equations could also be developed for the modified B–C ratio or for PW as the measure of equivalent worth.) Again, the *magnitude* of the B–C ratio will be different depending upon which approach is used to incorporate disbenefits, but project acceptability—that is, whether the B–C ratio is >, <, or = 1.0—will not be affected, as shown in Example 10-4.

Conventional B–C ratio with AW, *benefits* reduced by amount of *disbenefits*:

$$\text{B–C} = \frac{\text{AW(benefits)} - \text{AW(disbenefits)}}{\text{AW(costs)}} = \frac{\text{AW}(B) - \text{AW}(D)}{\text{CR} + \text{AW(O\&M)}}. \quad (10\text{-}5)$$

Here, AW(·) = annual worth of (·);
B = benefits of the proposed project;
D = disbenefits of the proposed project;
CR = capital recovery amount (i.e., the equivalent annual cost of the initial investment, I, including an allowance for market value, if any);
O&M = operating and maintenance costs of the proposed project.

Conventional B–C ratio with AW, *costs* increased by amount of *disbenefits*:

$$\text{B–C} = \frac{\text{AW(benefits)}}{\text{AW(costs)} + \text{AW(disbenefits)}} = \frac{\text{AW}(B)}{\text{CR} + \text{AW(O\&M)} + \text{AW}(D)}. \quad (10\text{-}6)$$

EXAMPLE 10-4 Including Disbenefits in a B–C Analysis

Refer back to Example 10-2. In addition to the benefits and costs, suppose that there are disbenefits associated with the runway extension project. Specifically, the increased noise level from commercial jet traffic will be a serious nuisance to homeowners living along the approach path to the Columbia Municipal Airport. The annual disbenefit to citizens of Columbia caused by this noise pollution is estimated to be \$100,000. Given this additional information, reapply the conventional B–C ratio, with equivalent annual worth, to determine whether this disbenefit affects your recommendation on the desirability of this project.

Solution

Disbenefits reduce benefits, Equation (10-5)	B–C = [AW(B) − AW(D)]/[CR + AW(O&M)] B–C = [\$490,000 − \$100,000]/[\$1,200,000 (A/P, 10%, 20) + \$197,500] **B–C = 1.152 > 1; extend runways.**
Disbenefits treated as additional costs, Equation (10-6)	B–C = AW(B)/[CR + AW(O&M) + AW(D)] B–C = \$490,000/[\$1,200,000 (A/P, 10%, 20) + \$197,500 + \$100,000] **B–C = 1.118 > 1; extend runways.**

As in the case of conventional and modified B–C ratios, the treatment of disbenefits may affect the magnitude of the B–C ratio, but it has no effect on project desirability in go/no-go decisions. It is left to the reader to develop a mathematical rationale for this, similar to that included in the discussion of conventional versus modified B–C ratios.

10.7.2 Added Benefits versus Reduced Costs in B–C Analyses

The analyst often needs to classify certain cash flows as either added benefits or reduced costs in calculating a B–C ratio. The questions arise, How critical is the proper assignment of a particular cash flow as an added benefit or a reduced cost? Is the outcome of the analysis affected by classifying a reduced cost as a benefit? *An arbitrary decision as to the classification of a benefit or a cost has no impact on project acceptability.* The mathematical rationale for this information is presented next and in Example 10-5.

Let B = the equivalent annual worth of project benefits,
C = the equivalent annual worth of project costs,
X = the equivalent annual worth of a cash flow (either an added benefit or a reduced cost) not included in either B or C.

If X is classified as an added benefit, *then* $\text{B–C} = \frac{B+X}{C}$. Alternatively,

if X is classified as a reduced cost, *then* $\text{B–C} = \frac{B}{C-X}$.

Assuming that the project is acceptable, that is, B–C ≥ 1.0,

$\frac{B+X}{C} \geq 1.0$, which indicates that $B + X \geq C$, and

$\frac{B}{C-X} \geq 1.0$, which indicates that $B \geq C - X$,

which can be restated as $B + X \geq C$.

EXAMPLE 10-5

Added Benefit versus Reduced Cost in a Bridge Widening Project

A project is being considered by the Tennessee Department of Transportation to replace an aging bridge across the Cumberland River on a state highway. The existing two-lane bridge is expensive to maintain and creates a traffic bottleneck because the state highway is four lanes wide on either side of the bridge. The new bridge can be constructed at a cost of \$300,000, and estimated annual maintenance costs are \$10,000. The existing bridge has annual maintenance costs of \$18,500. The annual benefit of the new four-lane bridge to motorists, due to the removal of the traffic bottleneck, has been estimated to be \$25,000. Conduct a B–C analysis, using a MARR of 8% and a study period of 25 years, to determine whether the new bridge should be constructed.

Solution

Treating the reduction in annual maintenance costs as a *reduced cost:*

$$\text{B–C} = \$25{,}000/[\$300{,}000(A/P, 8\%, 25) - (\$18{,}500 - \$10{,}000)]$$

$$\text{B–C} = 1.275 > 1; \text{ construct new bridge.}$$

Treating the reduction in annual maintenance costs as an *increased benefit:*

$$\text{B–C} = [\$25{,}000 + (\$18{,}500 - \$10{,}000)]/[\$300{,}000(A/P, 8\%, 25)]$$

$$\text{B–C} = 1.192 > 1; \text{ construct new bridge.}$$

Therefore, the decision to classify a cash-flow item as an additional benefit or as a reduced cost will affect the magnitude of the calculated B–C ratio, but it will have no effect on project acceptability.

10.8 Evaluating Independent Projects by B–C Ratios

Independent projects are categorized as groupings of projects for which the choice to select any particular project in the group is *independent* of choices regarding any and all other projects within the group. Thus, it is permissible to select none of the projects, any combination of projects, or all of the projects from an independent group. (Note that this does not hold true under conditions of *capital rationing*. Methods of evaluating otherwise independent projects under capital rationing are discussed in Chapter 13.) Because any or all projects from an independent set can be selected, formal comparisons of independent projects are unnecessary. The issue of whether one project is *better* than another is unimportant if those projects are independent; the only criterion for selecting each of those projects is whether their respective B–C ratios are equal to or greater than 1.0.

Mutually exclusive alternatives can be formed from independent projects. For instance, from three independent projects there are $2^3 = 8$ mutually exclusive combinations possible (including "do nothing"). See if you can list them all. For five independent projects, there are $2^5 = 32$ mutually exclusive alternatives.

A typical example of an economy study of a federal project—using the conventional B-C-ratio method—is the study of a flood control and power

project on the White River in Missouri and Arkansas. Considerable flooding and consequent damage had occurred along certain portions of this river, as shown in Table 10-2. In addition, the uncontrolled water flow increased flood conditions on the lower Mississippi River. In this case, there were independent options of building a reservoir *and/or* a channel improvement to alleviate the problem. The cost and benefit summaries for the Table Rock reservoir and the Bull Shoals channel improvement are shown in Table 10-3. The fact that the Bull Shoals

Annual Loss as a Result of Floods on Three Stretches of the White River

Item	Annual Value of Loss	Annual Loss per Acre of Improved Land in Floodplain	Annual Loss per Acre for Total Area in Floodplain
Crops	\$1,951,714	\$6.04	\$1.55
Farm (other than crops)	215,561	0.67	0.17
Railroads and highways	119,800	0.37	0.09
Levees[a]	87,234	0.27	0.07
Other losses	168,326	0.52	0.13
Total	\$2,542,635	\$7.87	\$2.01

[a] Expenditures by the United States for levee repairs and high-water maintenance.

Estimated Costs, Annual Charges, and Annual Benefits for the Table Rock Reservoir and Bull Shoals Channel Improvement Projects

Item	Table Rock Reservoir	Bull Shoals Channel Improvement
Cost of dam and appurtenances, and reservoir:		
Dam, including reservoir-clearing, camp, access railroads and highways, and foundation exploration and treatment	\$20,447,000	\$25,240,000
Powerhouse and equipment	6,700,000	6,650,000
Power-transmission facilities to existing load-distribution centers	3,400,000	4,387,000
Land	1,200,000	1,470,000
Highway relocations	2,700,000	140,000
Cemetery relocations	40,000	18,000
Damage to villages	6,000	94,500
Damage to miscellaneous structures	7,000	500
Total Construction Cost (estimated appropriation of public funds necessary for the execution of the project)	\$34,500,000	\$38,000,000
Federal investment:		
Total construction cost	\$34,500,000	\$38,000,000
Interest during construction	\$1,811,300	1,995,000
Total	\$36,311,300	\$39,995,000
Present value of federal properties	1,200	300
Total Federal Investment	\$36,312,500	\$39,995,300
Total Annual Costs	\$1,642,200	\$1,815,100

(Continued)

Estimated Costs, Annual Charges, and Annual Benefits for the Table Rock Reservoir and Bull Shoals Channel Improvement Projects (*continued*)

Item	Table Rock Reservoir	Bull Shoals Channel Improvement
Annual benefits:		
Prevented direct flood losses in White River basin:		
Present conditions	60,100	266,900
Future developments	19,000	84,200
Prevented indirect flood losses owing to floods in White River basin	19,800	87,800
Enhancement in property values in White River valley	7,700	34,000
Prevented flood losses on Mississippi River	220,000	980,000
Annual Flood Benefits	326,000	1,452,900
Power value	1,415,600	1,403,400
Total Annual Benefits	$1,742,200	$2,856,300
Conventional B–C Ratio = Total Annual Benefits ÷ Annual Costs	1.06	1.57

channel improvement project has the higher B–C ratio is irrelevant; both options are acceptable because their B–C ratios are greater than one.

Several interesting facts may be noted concerning this study. *First*, there was no attempt to allocate the cost of the projects between flood control and power production. *Second*, very large portions of the flood-control benefits were shown to be in connection with the Mississippi River and are not indicated in Table 10-2; these were not detailed in the main body of the report but were shown in an appendix. Only a moderate decrease in the value of these benefits would have changed the B–C ratio considerably. *Third*, without the combination of flood-control and power-generation objectives, neither project would have been economical for either purpose. These facts point to the advantages of multiple purposes for making flood-control projects economically feasible and to the necessity for careful enumeration and evaluation of the prospective benefits of a public-sector project.

10.9 Comparison of Mutually Exclusive Projects by B–C Ratios

Recall that a group of *mutually exclusive projects* was defined as *a group of projects from which, at most, one project may be selected.* When using an equivalent-worth method to select from among a set of mutually exclusive alternatives (MEAs), the best alternative can be selected by maximizing the PW (or AW, or FW). Because the B–C method provides a *ratio* of benefits to costs rather than a direct measure of each project's *profit potential*, selecting the project that maximizes the B–C ratio does not guarantee that the best project is selected. This phenomenon is illustrated in Example 10-6. As with the rate-of-return procedures in Chapter 6, an evaluation of mutually exclusive alternatives by the B–C ratio requires that an *incremental* B–C analysis be conducted.

EXAMPLE 10-6

Inconsistent Ranking Problem When B–C Ratios Are Inappropriately Compared

The required investments, annual operating and maintenance costs, and annual benefits for two mutually exclusive alternative projects are shown subsequently. Both conventional and modified B–C ratios are included for each project. Note that Project *A* has the greater *conventional* B–C, but Project *B* has the greater *modified* B–C. Given this information, which project should be selected?

	Project *A*	Project *B*
Capital investment	$110,000	$135,000
Annual O&M cost	12,500	45,000
Annual benefit	37,500	80,000
Conventional B–C	**1.475**	1.314
Modified B–C	1.934	**2.206**

MARR = 10% per year
Study period = 20 years

Solution

The B–C analysis has been conducted improperly. Although each of the B–C ratios shown is *numerically correct*, a comparison of mutually exclusive alternatives requires that an incremental analysis be conducted.

When comparing mutually exclusive alternatives with the B–C ratio method, *they are first ranked in order of increasing total equivalent worth of costs*. This rank-ordering will be identical whether the ranking is based on PW, AW, or FW of costs. The do-nothing alternative is selected as a baseline alternative. The B–C ratio is then calculated for the alternative having the lowest equivalent cost. If the B–C ratio for this alternative is equal to or greater than 1.0, then that alternative becomes the new baseline; otherwise, do-nothing remains as the baseline. The next least equivalent cost alternative is then selected, and the difference (Δ) in the respective benefits and costs of this alternative and the baseline is used to calculate an incremental B–C ratio ($\Delta B/\Delta C$). If that ratio is equal to or greater than 1.0, then the higher equivalent cost alternative becomes the new baseline; otherwise, the last baseline alternative is maintained. Incremental B–C ratios are determined for each successively higher equivalent cost alternative until the last alternative has been compared. The flowchart of this procedure is included as Figure 10-3, and the procedure is illustrated in Example 10-7.

EXAMPLE 10-7

Incremental B–C Analysis of Mutually Exclusive Projects

Three mutually exclusive alternative public-works projects are currently under consideration. Their respective costs and benefits are included in the table that follows. Each of the projects has a useful life of 50 years, and MARR is 10% per

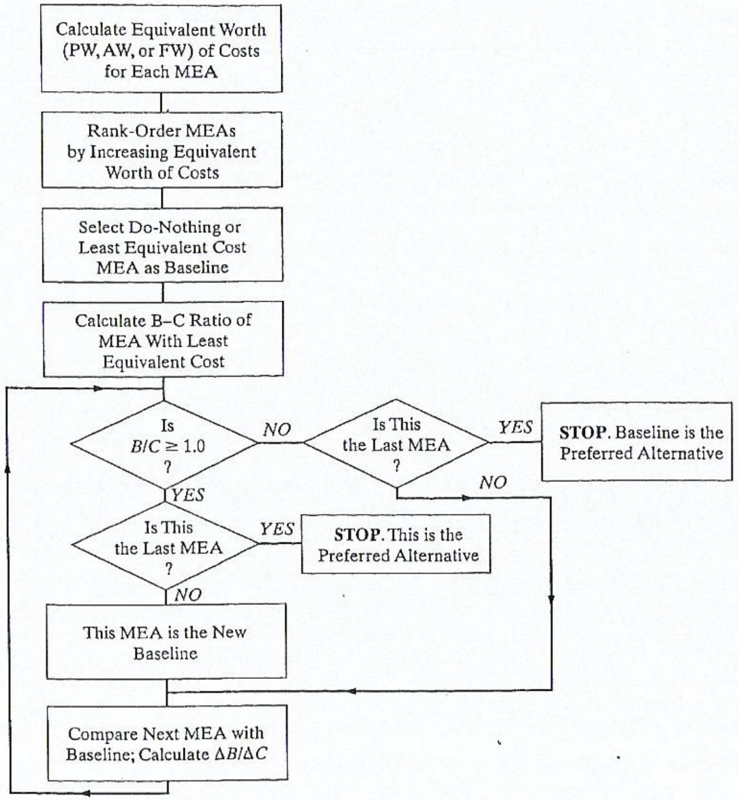


Figure 10-3 The Incremental B–C Ratio Procedure

year. Which, if any, of these projects should be selected? Solve by hand and by spreadsheet.

	A	*B*	*C*
Capital investment	$8,500,000	$10,000,000	$12,000,000
Annual operating and maintenance costs	750,000	725,000	700,000
Market value	1,250,000	1,750,000	2,000,000
Annual benefit	2,150,000	2,265,000	2,500,000

Solution by Hand

$$\text{PW(Costs, } A) = \$8{,}500{,}000 + \$750{,}000(P/A, 10\%, 50) - \$1{,}250{,}000(P/F, 10\%, 50) = \$15{,}925{,}463,$$
$$\text{PW(Costs, } B) = \$10{,}000{,}000 + \$725{,}000(P/A, 10\%, 50) - \$1{,}750{,}000(P/F, 10\%, 50) = \$17{,}173{,}333,$$
$$\text{PW(Costs, } C) = \$12{,}000{,}000 + \$700{,}000(P/A, 10\%, 50) - \$2{,}000{,}000(P/F, 10\%, 50) = \$18{,}923{,}333,$$
$$\text{PW(Benefit, } A) = \$2{,}150{,}000(P/A, 10\%, 50) = \$21{,}316{,}851,$$
$$\text{PW(Benefit, } B) = \$2{,}265{,}000(P/A, 10\%, 50) = \$22{,}457{,}055,$$
$$\text{PW(Benefit, } C) = \$2{,}750{,}000(P/A, 10\%, 50) = \$24{,}787{,}036.$$
$$\text{B–C}(A) = \$21{,}316{,}851/\$15{,}925{,}463 = 1.3385 > 1.0.$$

Therefore, Project A is acceptable.

$$\Delta B/\Delta C \text{ of } (B - A) = (\$22{,}457{,}055 - \$21{,}316{,}851)/(\$17{,}173{,}333 - \$15{,}925{,}463) = 0.9137 < 1.0.$$

Therefore, increment required for Project B is not acceptable.

$$\Delta B/\Delta C \text{ of } (C - A) = (\$24{,}787{,}036 - \$21{,}316{,}851)/(\$18{,}923{,}333 - \$15{,}925{,}463) = 1.1576 > 1.0.$$

Therefore, increment required for Project C is acceptable.

Decision: *Recommend Project C.*

Spreadsheet Solution

A spreadsheet analysis for this example is shown in Figure 10-4. For each mutually exclusive project, the PW of total benefits (cells B11:B13) and the PW of total costs (cells C11:C13) are calculated. Note that annual operating and maintenance costs are included in the total cost calculation in accordance with the conventional B–C ratio formulation.

Since the projects are mutually exclusive, they must be ranked from smallest to largest according to the equivalent worth of costs (do nothing → A → B → C). The B–C ratio for Project A is calculated to be 1.34 (row 17), and Project A replaces do-nothing as the baseline alternative.

Project B is now compared with Project A (row 18). The ratio of incremental benefits to incremental costs is less than 1.0, so Project A remains the baseline alternative. Finally, in row 19, the incremental benefits and incremental costs

= B5 + PV(B2, B1, -C5) - E5 / (1 + B2)^B1

	A	B	C	D	E
1	Useful Life =	50			
2	MARR =	10%			
3					
4	Project	Capital Investment	Ann. Maint Cost	Annual Benefit	Market Value
5	A	$8,500,000	$750,000	$2,150,000	$ 1,250,000
6	B	$10,000,000	$725,000	$2,265,000	$ 1,750,000
7	C	$12,000,000	$700,000	$2,500,000	$ 2,000,000
8					
9		Present Worth			
10	Project	Benefits	Costs		
11	A	$ 21,316,851	$ 15,925,463		
12	B	$ 22,457,055	$ 17,173,333		
13	C	$ 24,787,036	$ 18,923,333		
14					
15	Present Worth Formulation				
16		Inc. Benefit	Inc. Cost	Inc. B/C	
17	0 -> A	$ 21,316,851	$ 15,925,463	1.34	
18	A -> B	$ 1,140,204	$ 1,247,870	0.91	
19	A -> C	$ 3,470,185	$ 2,997,870	1.16	

= PV(B2, B1, –D5)

= B11

= B12 – B11

= B13 – B11

= B17 / C17

Figure 10-4 Spreadsheet Solution, Example 10-7

associated with selecting Project C instead of Project A are used to calculate an incremental B–C ratio of 1.16. Since this ratio is greater than one, the increment required for Project C is acceptable, and thus Project C becomes the recommended project.

It is not uncommon for some of the projects in a set of mutually exclusive public-works projects to have different lives. Recall from Chapter 6 that the AW criterion can be used to select from among alternatives with different lives as long as the assumption of *repeatability* is valid. Similarly, if a mutually exclusive set of public-works projects includes projects with varying useful lives, it may be possible to conduct an incremental B–C analysis by using the AW of benefits and costs of the various projects. This analysis is illustrated in Example 10-8.

EXAMPLE 10-8 **B–C Analysis with Unequal Project Lives**

Two mutually exclusive alternative public-works projects are under consideration. Their respective costs and benefits are included in the table that follows. Project *I* has an anticipated life of 35 years, and the useful life of Project *II* has been estimated to be 25 years. If the MARR is 9% per year, which, if either, of these projects should be selected? The effect of inflation is negligible.

	Project *I*	Project *II*
Capital investment	$750,000	$625,000
Annual operating and maintenance	120,000	110,000
Annual benefit	245,000	230,000
Useful life of project (years)	35	25

Solution

$$\text{AW(Costs}, I) = \$750{,}000(A/P, 9\%, 35) + \$120{,}000 = \$190{,}977,$$

$$\text{AW(Costs}, II) = \$625{,}000(A/P, 9\%, 25) + \$110{,}000 = \$173{,}629,$$

$$\text{B–C}(II) = \$230{,}000/\$173{,}629 = 1.3247 > 1.0.$$

Therefore, Project II is acceptable.

$$\Delta B/\Delta C \text{ of } (I\text{–}II) = (\$245{,}000 - \$230{,}000)/(\$190{,}977 - \$173{,}629)$$
$$= 0.8647 < 1.0.$$

Therefore, increment required for Project *I* is not acceptable.

Decision: *Project II should be selected.*

10.10 CASE STUDY—Improving a Railroad Crossing

Traffic congestion and vehicle safety are significant concerns in most major cities in the Northeast United States. A major metropolitan city in New Jersey is considering the elimination of a railroad grade crossing by building an overpass. Traffic engineers estimated that approximately 2,000 vehicles per day are delayed at an average of 2 minutes each due to trains at the grade crossing. Trucks comprise 40% of the vehicles, and the opportunity cost of their delay is assumed to average $20 per truck-hour. The other vehicles are cars having an assumed average opportunity cost of $4 per car-hour. It is also estimated that the new overpass will save the city approximately $4,000 per year in expenses directly due to accidents.

The traffic engineers determined that the overpass would cost $1,000,000 and is estimated to have a useful life of 40 years and a $100,000 salvage value. Annual maintenance costs of the overpass would cost the city $5,000 more than the maintenance costs of the existing grade crossing. The installation of the overpass will save the railroad an annual expense of $30,000 for lawsuits and maintenance of crossing guards.

Since this is a public project, there are special considerations and a complete and comprehensive engineering economy study is more challenging than in the case of privately financed projects. For example, in the private sector, costs are accrued by the firm undertaking the project, and benefits are the favorable outcomes achieved by the firm. Typically, any costs and benefits that are external to the firm are ignored in economic evaluations unless those external costs and benefits indirectly affect the firm. With economic evaluations of public projects, however, the opposite is true. As in the case of improving the railroad crossing, there are multiple purposes or objectives to consider. The true owners of the project are the taxpayers! The monetary impacts of the diverse benefits are oftentimes hard to quantify, and there may be special political or legal issues to consider.

In this case study, the city council is now in the process of considering the merits of the engineering proposal to improve the railroad crossing. The city council is considering the following questions in its deliberations:

- Should the overpass be built by the city if it is to be the owner and the opportunity cost of the city's capital is 8% per year?
- How much should the railroad reasonably be asked to contribute toward construction of the bridge if its opportunity cost of capital is assumed to be 15% per year?

Solution

The city uses the conventional B–C ratio with AW for its analyses of public projects. The annual benefits of the overpass are comprised of the time savings for vehicles (whose drivers are "members" of the city) and the reduction in accident expenses. The city's cost engineer makes the following estimates.

Annual Benefits:

$$\text{Cars} = \left(\frac{2{,}000 \times 0.6 \text{ vehicles}}{\text{day}}\right)\left(\frac{365 \text{ days}}{\text{year}}\right)\left(\frac{\text{hour}}{60 \text{ minutes}}\right)\left(\frac{\$4.00}{\text{car} - \text{hour}}\right) = \$58{,}400$$

$$\text{Trucks} = \left(\frac{2{,}000 \times 0.4 \text{ vehicles}}{\text{day}}\right)\left(\frac{365 \text{ days}}{\text{year}}\right)\left(\frac{\text{hour}}{60 \text{ minutes}}\right)\left(\frac{\$20.00}{\text{truck} - \text{hour}}\right)$$

$$= \$194{,}667$$

$$\text{Annual savings} = \$4{,}000$$

$$\text{Total annual benefits} = \$58{,}400 + \$194{,}667 + \$4{,}000 = \$257{,}067.$$

Notice that the estimated $30,000 annual expense savings for lawsuits and maintenance of the crossing guard is not included in the annual benefits calculation. This savings will be experienced by the owners of the railroad, not by the city.

The costs of the overpass to the city are the construction of the overpass (less its salvage value) and the increased maintenance costs. The cost engineer makes the following estimates.

Annual Costs:

$$\text{Capital recovery} = \$1{,}000{,}000(A/P, 8\%, 40) - \$100{,}000(A/F, 8\%, 40) = \$83{,}474$$

$$\text{Increased maintenance} = \$5{,}000$$

$$\text{Total annual costs} = \$83{,}474 + \$5{,}000 = \$88{,}474.$$

Based on these estimates, the B–C ratio of the proposed overpass is

$$\text{B–C Ratio} = \frac{\text{Annual benefits}}{\text{Annual costs}} = \frac{\$257{,}067}{\$88{,}474} = 2.91.$$

The cost engineer recommends to the city council that the new overpass be built since the B–C ratio is greater than 1.0.

The cost engineer also advises the council that since the railroad company stands to directly benefit from the replacement of the existing grade crossing by the overpass, it would not be unreasonable for the city to request a contribution to the construction cost of the overpass. Given the railroad company's cost of capital of 15% per year and an estimated annual savings of $30,000, the cost engineer calculates that the overpass is worth

$$\text{PW}(15\%) = \$30{,}000(P/A, 15\%, 40) = \$199{,}254$$

to the railroad. Any amount contributed by the railroad company would serve to reduce the denominator of the B–C ratio, thereby increasing the value of the ratio. The B–C ratio was computed to be 2.91 without any contribution by the railroad. Therefore, the cost engineer concludes that the city should plan on constructing the overpass regardless of whether or not the railroad company can be persuaded to contribute financially to the project.

This case illustrates some of the special issues associated with providing economic evaluations of public-sector projects. While the B–C ratio is a useful method for evaluating projected financial performance of a public-sector project, the quantification of benefits and costs may prove difficult. As the case illustrates, typically, surrogate or proxy measures are used in the estimation of benefits and costs. Also, it is especially important to remember the perspectives of the owners in the evaluation of public projects—not necessarily the elected city council members, but the taxpayers!

10.11 Summary

From the discussion and examples of public projects presented in this chapter, it is apparent that, because of the methods of financing, the absence of tax and profit requirements, and political and social factors, the criteria used in evaluating privately financed projects frequently cannot be applied to public works. Nor should public projects be used as yardsticks with which to compare private projects. Nevertheless, whenever possible, public works should be justified on an economic basis to ensure that the public obtains the maximum return from the tax money that is spent. Whether an engineer is working on such projects, is called upon to serve as a consultant, or assists in the conduct of the B–C analysis, he or she is bound by professional ethics to do his or her utmost to see that the projects and the associated analyses are carried out in the best possible manner and within the limitations of the legislation enacted for their authorization.

The B–C ratio has remained a popular method for evaluating the financial performance of public projects. Both the conventional and modified B–C ratio methods have been explained and illustrated for the case of *independent* and *mutually exclusive* projects. A final note of caution: The best project among a *mutually* *exclusive* set of projects is not necessarily the one that maximizes the B–C ratio. In this chapter, we have seen that an incremental analysis approach to evaluating benefits and costs is necessary to ensure the correct choice.

Problems

The number in parentheses that follows each problem refers to the section from which the problem is taken.

10-1. A wastewater treatment plant is expected to require an initial investment of $500,000 and annual maintenance expenses of $80,000. The benefits to the public are valued at $100,000 per year. This project can be assumed to have an infinite life. If MARR is 10% per year, determine whether the project is economically attractive the B–C ratio measure of merit. **(10.7)**

10-2. A dam construction company has $2 billion available for investment. Project *A* yields an estimated $0.5 million per year in benefits to the public on an investment of $1.5 billion and annual maintenance expenses of $5 million. Project *B* requires $3 billion, with maintenance expenses of $8 million per year and is expected to produce annual savings of $6 million. Both projects are expected to last 30 years. For a MARR of 10%, compute the benefit–cost ratio for each project and make a recommendation. **(10.7)**

10-3. A proposal has been made for improving the downtown area of a small town. The plan calls for banning vehicular traffic on the main street and turning this street into a pedestrian mall with tree plantings and other beautification features. This plan will involve actual costs of $7,000,000 and, according to its proponents, the plan will produce benefits and disbenefits to the town as follows:

Benefits:

Increased sales tax revenue	$400,000 per year
Increased real estate property taxes	$300,000 per year
Benefits due to decreased air pollution	$50,000 per year
Quality of life improvements to users	$50,000 per year

Disbenefits:

Increased maintenance $150,000 per year

a. Compute the B–C ratio of this plan based on a MARR of 8% per year and an infinite life for the project. **(10.7)**

b. How does the B–C ratio change for a 20-year project life and a MARR of 10% per year? **(10.7)**

10-4. A retrofitted space-heating system is being considered for a small office building. The system can be purchased and installed for $120,000, and it will save an estimated 300,000 kilowatt-hours (kWh) of electric power each year over a six-year period. A kilowatt-hour of electricity costs $0.10, and the company uses a MARR of 15% per year in its economic evaluations of refurbished systems. The market value of the system will be $8,000 at the end of six years, and additional annual operating and maintenance expenses are negligible. Use the benefit–cost method to make a recommendation. **(10.7)**

10-5. In Problem 10-4, what is the benefit–cost ratio of the project if the general inflation rate is 4% per year and the market value is negligible? The market interest rate (i_m) is 18% per year, and the annual savings are expressed in year-zero dollars. **(10.7, Chapter 8)**

10-6. A city is considering buying a piece of land for $500,000 and constructing an office complex on it. Their planning horizon is 20 years. Three mutually exclusive building designs (shown below) have been drawn up by an architectural firm. Use the modified benefit-cost ratio method and a MARR of 10% per year to determine which alternative, if any, should be recommended to the city council. **(10.9)**

	Design *A*	Design *B*	Design *C*
Cost of the building, including cost of the land	$1,300,000	$1,700,000	$3,500,000
Resale value of land and building at end of 20-year planning horizon	$500,000	$900,000	$2,000,000
Annual net rental income (after deducting all operating expenses)	$120,000	$300,000	$450,000

10-7. A toll bridge across the Mississippi River is being considered as a replacement for the current I-40 bridge linking Tennessee to Arkansas. Because this bridge, if approved, will become a part of the U.S. Interstate Highway system, the B–C ratio method must be applied in the evaluation. Investment costs of the structure are estimated to be $17,500,000, and $325,000 per year in operating and maintenance costs are anticipated. In addition, the bridge must be resurfaced every fifth year of its 30-year projected life at a cost of $1,250,000 per occurrence (no resurfacing cost in year 30). Revenues generated from the toll are anticipated to be $2,500,000 in its first year of operation, with a projected annual rate of increase of 2.25% per year due to the anticipated annual increase in traffic across the bridge. Assuming zero market (salvage) value for the bridge at the end of 30 years and a MARR of 10% per year, should the toll bridge be constructed? **(10.7)**

10-8. Refer back to Problem 10-7. Suppose that the toll bridge can be redesigned such that it will have a (virtually) infinite life. MARR remains at 10% per year. Revised costs and revenues (benefits) are given as follows: **(10.7, 10.9)**

Capital investment: $22,500,000
Annual operating and maintenance costs: $250,000
Resurface cost every seventh year: $1,000,000
Structural repair cost, every 20th year: $1,750,000
Revenues (treated as constant—no rate of increase): $3,000,000

a. What is the capitalized worth of the bridge?

b. Determine the B–C ratio of the bridge over an infinite time horizon.

c. Should the initial design (*Problem 10-7*) or the new design be selected?

10-9. The government is planning a multiple-purpose hydroelectric project for a river basin. The estimated benefits and costs that are expected to be derived from the three alternatives under consideration are as follows: **(10.8)**

The interest rate is 8% and the life of each project is 40 years.

a. Using benefit–cost analysis, determine which project should be selected.

b. What is the rank-ordering of projects from best to worst?

c. If the projects involved intangible benefits that required considerable judgment in assigning their values, would your recommendation be affected?

Initial Cost	*A*	*B*	*C*
Annual benefits and costs	$30,000,000	$40,000,000	$50,000,000
Power sales	$1,600,000	$1,250,000	$1,800,000
Flood control saving	$225,000	$300,000	$500,000
Irrigation benefits	$8,000,000	$450,000	$600,000
Recreation benefits	$525,000	$150,000	$350,000
Operating and maintenance costs	$220,000	$230,000	$350,000

10-10. Three mutually exclusive projects are being considered for investment. Their costs and their estimated benefits are as follows: **(10.8)**

	PW ($)	
Project	Costs	Benefits
A	$115,000	$135,000
B	$230,000	$255,000
C	$80,000	$120,000

a. Which plan(s) should be adopted, if any, if the controlling board wishes to invest any amount required, provided that the B–C ratio on the required investment is at least 1?

b. Suppose that 8% of the costs of each plan are reclassified as disbenefits. What percentage change in the B–C ratio of each plan results from the reclassification?

c. Comment on why the rank-orderings in (a) are unaffected by the change in (b).

10-11. The government is considering five mutually exclusive proposals for the improvement of the postal service. The annual equivalent costs and estimated benefits of alternatives are as follows:

	Annual Equivalent ($)	
Alternative	Costs	Benefits
A	$1,200	$1,000
B	$2,000	$2,400
C	$2,600	$2,800
D	$4,000	$4,500
E	$5,000	$5,000

Which proposal, if any, should be adopted if the postal department wishes to invest if, and only if, the B-C ratiois at least 1? (10.9)

10-12. The civic amenities department of the federal government is planning on investing in the drainage system of a new township. Four mutually exclusive systems have been proposed, and their capital investment costs and net annual benefits are the following (salvage values are negligible):

EOY	A	B	C	D
0	−$200	−$250	−$300	−$280
1	$100	$130	$140	$150
2	$80	$125	$120	$125
3	$70	$90	$100	$130
4	$60	$70	$125	$110

If the town's MARR is 12% per year, use the B–C ratio method to determine which proposal is the best. (10.9)

10-13. A nonprofit government corporation is considering two alternatives for generating power:

Alternative A. Build a coal-powered generating facility at a cost of $20,000,000. Annual power sales are expected to be $1,000,000 per year. Annual operating and maintenance costs are $200,000 per year. A benefit of this alternative is that it is expected to attract new industry, worth $500,000 per year, to the region.

Alternative B. Build a hydroelectric generating facility. The capital investment, power sales, and operating costs are $30,000,000, $800,000, and $100,000 per year, respectively. Annual benefits of this alternative are as follows:

Flood-control savings	$600,000
Irrigation	$200,000
Recreation	$100,000
Ability to attract new industry	$400,000

The useful life of both alternatives is 50 years. Using an interest rate of 5%, determine which alternative (if either) should be selected according to the conventional B–C-ratio method. (10.9)

10-14. A new storm drainage system must be constructed right away to reduce periodic flooding that occurs in a city that is in a valley. Five mutually exclusive designs have been proposed, and their present worth (in thousands of dollars) of costs and benefits are the following. (10.9)

	System				
	1	2	3	4	5
PW of costs	$1,000	$4,000	$4,000	$10,000	$12,000
PW of benefits	8,000	8,000	14,000	16,000	24,000

a. Which system has the greatest B–C ratio?

b. Which system has the largest incremental B–C ratio (based on differences between alternatives)?

c. Which system should be chosen?

10-15. Refer to Example 10-7. At a public hearing, someone argued that the market values cannot be adequately estimated 50 years into the future. How is the decision changed if the market values are treated as $0 for all three alternatives? (10.9)

10-16. Consider the mutually exclusive alternatives in Table P10-16. Which alternative would be chosen according to these decision criteria?

a. Maximum benefit

b. Minimum cost

c. Maximum benefits minus costs

d. Largest investment having an incremental B–C ratio larger than one

e. Largest B–C ratio

Which project should be chosen? (10.9)

10-17. Three mutually exclusive projects are being considered for a new park. The life of the park is expected to be 100 years and the sponsoring agency's MARR is 10% per year. Annual benefits to the public have been estimated by a committee and are shown below. Use the B–C method (incrementally) to select the best project. (10.9)

Mutually Exclusive Alternatives for Problem 10-16

Alternative	Equivalent Annual Cost of Project	Expected Annual Flood Damage	Annual Benefits
I. No flood control	0	$100,000	0
II. Construct levees	$30,000	80,000	$112,000
III. Build small dam	$100,000	5,000	110,000

Mutually Exclusive Plans for Problem 10-19

Route	Construction Costs	Annual Maintenance Cost	Annual Savings in Fire Damage	Annual Recreational Benefit	Annual Timber Access Benefit
A	$185,000	$2,000	$5,000	$3,000	$500
B	220,000	3,000	7,000	6,500	1,500
C	290,000	4,000	12,000	6,000	2,800

	Alternative		
	A	B	C
Initial costs	$500,000	$550,000	$600,000
Annual benefits	$80,000	$90,000	$95,000
Benefit cost ratio	1.6	1.63	1.583

10-18. The local council of a suburban community is considering three mutually exclusive projects. Their respective costs and benefits are given in the table below. Each of the projects has useful life of 60 years and the nominal rate of interest is 10% per year. Which of these three projects should be selected? (10.9)

	A	B	C
Capital investments	$800,000	$1,200,000	$1,400,000
Annual O&M costs	$75,000	$78,000	$68,000
Salvage value	$100,000	$125,000	$150,000
Annual benefit	$200,000	$225,000	$230,000

10-19. A state-sponsored Forest Management Bureau is evaluating alternative routes for a new road into a formerly inaccessible region. Three mutually exclusive plans for routing the road provide different benefits, as indicated in Table P10-19. The roads are assumed to have an economic life of 50 years, and MARR is 8% per year. Which route should be selected according to the B–C ratio method? (10.9)

10-20. The city of Oak Ridge is evaluating three mutually exclusive landscaping plans for refurbishing a public greenway. Benefits to the community have been estimated by a landscaping committee, and the costs of planting trees and shrubbery, as well as maintaining the greenway, are summarized below. The city's discount rate is 8% per year, and the planning horizon is 10 years. (10.9)

	Landscaping Plan		
	A	B	C
Initial planting cost	$75,000	$50,000	$65,000
Annual maintenance expense	4,000	5,000	4,700
Annual community benefits	20,000	18,000	20,000

a. Use the B–C ratio method to recommend the best plan when annual maintenance expenses offset annual benefits in the numerator.

b. Repeat Part (a) when annual maintenance expenses add to total costs in the denominator. Which plan is best?

c. Should the recommendations in Part (a) and (b) be the same? Why or why not?

10-21. An area on the Colorado River is subject to periodic flood damage that occurs, on the average, every two years and results in a $2,000,000 loss. It has been proposed that the river channel be straightened and deepened, at a cost of $2,500,000, to reduce the probable damage to not over $1,600,000 for each occurrence during a period of 20 years before it would have to be deepened again. This procedure would also involve annual expenditures of $80,000 for minimal maintenance. One legislator in the area has proposed that a better solution would be to construct a flood-control dam at a cost of $8,500,000, which would last indefinitely, with annual maintenance costs of not over $50,000. He estimates that this project would reduce the probable annual flood damage to not over $450,000. In addition, this solution would provide a substantial amount of irrigation water that would produce annual revenue of $175,000 and recreational facilities, which he estimates would be worth at least $45,000 per year to the adjacent populace. A second legislator believes that the dam should be built and that the river channel also should be straightened and deepened, noting that the total cost of $11,000,000 would reduce the probable annual flood loss to not over $350,000 while providing the same irrigation and recreational benefits. If the state's capital is worth 10%, determine the B–C ratios and the incremental B–C ratio. Recommend which alternative should be adopted. **(10.9)**

10-22. Ten years ago, the port of Secoma built a new pier containing a large amount of steel work, at a cost of $300,000, estimating that it would have a life of 50 years. The annual maintenance cost, much of it for painting and repair caused by environmental damage, has turned out to be unexpectedly high, averaging $27,000. The port manager has proposed to the port commission that this pier be replaced immediately with a reinforced concrete pier at a construction cost of $600,000. He assures them that this pier will have a life of at least 50 years, with annual maintenance costs of not over $2,000. He presents the information in Table P10-22 as justification for the replacement, having determined that the net market value of the existing pier is $40,000.

He has stated that, because the port earns a net profit of over $3,000,000 per year, the project could be financed out of annual earnings. Thus, there would be no interest cost, and an annual savings of $19,000 would be obtained by making the replacement. **(10.9)**

a. Comment on the port manager's analysis.

b. Make your own analysis and recommendation regarding the proposal.

10-23. You have been requested to recommend one of the mutually exclusive industrial sanitation control systems that are given below. If MARR is 15% per year, which system would you select? Use the B–C method. **(10.9)**

	Alternative	
	Gravity-fed	Vacuum-led
Capital investment	$24,500	$37,900
Annual receipts less expenses	8,000	8,000
Life in years	5	10

10-24. In the aftermath of Hurricane Thelma, the U.S. Army Corps of Engineers is considering two alternative approaches to protect a freshwater wetland from the encroaching seawater during high tides. The first alternative, the construction of a 5-mile long, 20-foot-high levee, would have an investment cost of $25,000,000 with annual upkeep costs estimated at $725,000. A new roadway along the top of the levee would provide two major benefits: (1) improved recreational access for fishermen and (2) reduction of the driving distance between the towns at opposite ends of the proposed levee by 11 miles. The annual benefit for the levee has been estimated at $1,500,000. The second alternative, a channel-dredging operation, would have an investment cost of $15,000,000. The annual cost of maintaining the channel is estimated at $375,000. There are no documented benefits for the channel-dredging project. Using a MARR of 8% and assuming a 25-year life for either alternative, apply the incremental B–C ratio ($\Delta B/\Delta C$) method to determine which alternative should be chosen. (Note: The null alternative, Do nothing, is not a viable alternative.) **(10.9)**

Pier Replacement Cost for Problem 10-22

Annual Cost of Present Pier		Annual Cost of Proposed Pier	
Depreciation ($300,000/50)	$6,000	Depreciation ($600,000/50)	$12,000
Maintenance cost	27,000	Maintenance cost	2,000
Total	$33,000	Total	$14,000

10-25. *Extended Learning Exercise* The Fox River is bordered on the east by Illinois Route 25 and on the west by Illinois Route 31. Along one stretch of the river, there is a distance of 16 miles between adjacent crossings. An additional crossing in this area has been proposed, and three alternative bridge designs are under consideration. Two of the designs have 25-year useful lives, and the third has a useful life of 35 years. Each bridge must be resurfaced periodically, and the roadbed of each bridge will be replaced at the end of its useful life, at a cost significantly less than initial construction costs. The annual benefits of each design differ on the basis of disruption to normal traffic flow along Routes 25 and 31. Given the information in Table P10-25, use the B–C ratio method to determine which bridge design should be selected. Assume that the selected design will be used indefinitely, and use a MARR of 10% per year. **(10.9)**

Bridge Design Information for Problem 10-25

	Bridge Design		
	A	*B*	*C*
Capital investment	$17,000,000	$14,000,000	$12,500,000
Annual maintenance cost*	12,000	17,500	20,000
Resurface (every fifth year)*	–	40,000	40,000
Resurface (every seventh year)*	40,000	–	–
Bridge replacement cost	3,000,000	3,500,000	3,750,000
Annual benefit	2,150,000	1,900,000	1,750,000
Useful life of bridge (years)**	35	25	25

* Cost not incurred in last year of bridge's useful life.
** Applies to roadbed only; structural portion of bridge has indefinite useful life.

FE Practice Problems

The city council of Morristown is considering the purchase of one new fire truck. The options are Truck X and Truck Y. The appropriate financial data are as follows:

	Truck *X*	Truck *Y*
Capital investment	$50,000	$60,000
Maintenance cost per year	$6,000	$4,500
Useful life	6 years	6 years
Reduction in fire damage per year	$18,000	$20,000

The purchase is to be financed by money borrowed at 12% per year. Use this information to answer Problems 10-26 and 10-27.

10-26. What is the conventional B–C ratio for Truck X? **(10.7)**

(a) 1.048 (b) 0.87 (c) 1.64 (d) 1.10 (e) 1.15

10-27. Which fire truck should be purchased? **(10.9)**

(a) Neither Truck X nor Truck Y

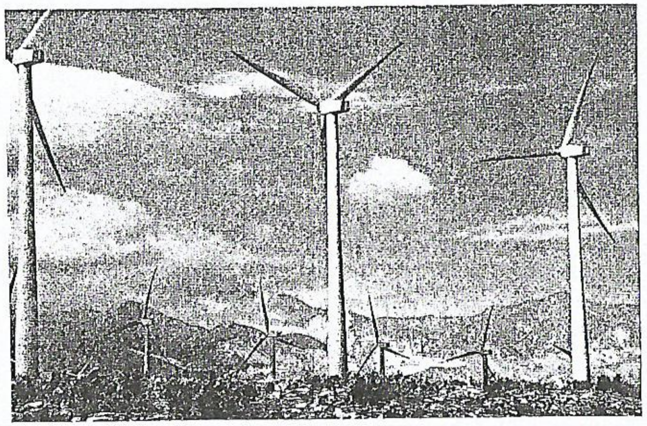
(b) Truck X

(c) Truck Y

(d) Both Truck X and Truck Y

CHAPTER 12

Probabilistic Risk Analysis



The aim of Chapter 12 is to discuss and illustrate several probabilistic methods that are useful in analyzing risk and uncertainty associated with engineering economy studies.

The Risks of Global Warming*

Risk is a condition where there is a possibility of adverse deviation from a desired and expected outcome. The risks of global climate change caused by carbon dioxide and other greenhouse gases include heightened regulation, revenue loss, and increased physical property impairment. Opportunities for mitigating the risks associated with climate change are numerous: increased efficiency of energy production and use, improved agricultural practices, and carbon capture and sequestration are just a few of the choices we have. In this chapter, you will learn how various probabilistic techniques can be used to assess the risks of engineering projects such as those that mitigate the effects of global warming.

* *Financial Times*, October 31, 2006, 1–2.

Use every possible means . . . to come with certainty at the enemy's strength, situation and movements—without this we wander in a wilderness of uncertainties.

—George Washington (1776)

12.1 Introduction

In previous chapters, we stated specific assumptions concerning applicable revenues, costs, and other quantities important to an engineering economy analysis. It was assumed that a high degree of confidence could be placed in all estimated values. That degree of confidence is sometimes called *assumed certainty*. Decisions made solely on the basis of this kind of analysis are sometimes called *decisions under certainty*. The term is rather misleading in that there rarely is a case in which the best of estimates of quantities can be assumed as certain.

We now consider the more realistic situation in which estimated future quantities are uncertain and project outcomes are risky. The motivation for dealing with risk and uncertainty is to establish the bounds of error such that another alternative being considered may turn out to be a better choice than the one we recommended under assumed certainty.

Both *risk* and *uncertainty* in decision-making activities are caused by lack of precise knowledge regarding future business conditions, technological developments, synergies among funded projects, and so on. *Decisions under risk* are decisions in which the analyst models the decision problem in terms of assumed possible future outcomes, or scenarios, whose probabilities of occurrence can be estimated. A *decision under uncertainty*, by contrast, is a decision problem characterized by several unknown futures for which probabilities of occurrence cannot be estimated. In reality, the difference between risk and uncertainty is somewhat arbitrary.

The probability that a cost, revenue, useful life, or other factor value will occur, or that a particular equivalent-worth or rate-of-return value for a cash flow will occur, is usually considered to be the long-run relative frequency with which the event (value) occurs or the subjectively estimated likelihood that it will occur. Factors such as these, having probabilistic outcomes, are called *random variables*. For example, the cash-flow amounts for an alternative, such as CO_2 capture and sequestration at a power plant, often result from the sum, difference, product, or quotient of random variables. In such cases, the measures of profitability (e.g., equivalent-worth and rate-of-return values) of an alternative's cash flows will also be random variables.

The information about these random variables, which is particularly helpful in decision making, is their expected values and variances, especially for the economic measures of merit of the alternatives. These derived quantities for the random variables are used to make the uncertainty associated with each alternative more explicit, including any probability of loss. Thus, when uncertainty is considered, the variability in the economic measures of merit and the probability of loss associated with the alternatives are normally used in the decision-making process.

12.2 Sources of Uncertainty

It is useful to consider some of the factors that affect the uncertainty involved in the analysis of the future economic consequences of an engineering project. It would be almost impossible to list and discuss all of the potential factors. There are four major sources of uncertainty, however, that are nearly always present in engineering economy studies.

The first source that is always present is the *possible inaccuracy of the cash-flow estimates used in the study*. The accuracy of the cash-inflow estimates is difficult to determine. If they are based on past experience or have been determined by adequate market surveys, a fair degree of reliance may be placed on them. On the other hand, if they are based on limited information with a considerable element of hope thrown in, they probably contain a sizable element of uncertainty.

The second major source affecting uncertainty is the *type of business involved in relation to the future health of the economy*. Some types of business operations are less stable than others. For example, most mining enterprises are more risky than one engaged in manufactured homes. Whenever capital is to be invested in an engineering project, the nature of the business as well as expectations of future economic conditions (e.g., interest rates) should be considered in deciding what risk is present.

A third source affecting uncertainty is the *type of physical plant and equipment involved*. Some types of structures and equipment have rather definite lives and market values. A good engine lathe generally can be used for many purposes in nearly any fabrication shop. Quite different would be a special type of lathe that was built to do only one unusual job. Its value would be dependent almost entirely upon the demand for the special task that it can perform. Thus, the type of physical property involved affects the accuracy of the estimated cash-flow patterns.

The fourth important source of uncertainty that must always be considered is the *length of the study period* used in the analysis. A long study period naturally decreases the probability of all the factors turning out as estimated. Therefore, a long study period, all else being equal, generally increases the uncertainty of a capital investment.

12.3 The Distribution of Random Variables

Capital letters such as X, Y, and Z are usually used to represent random variables and lowercase letters (x, y, z) to denote the particular values that these variables take on in the sample space (i.e., in the set of all possible outcomes for each variable). When a random variable's sample space is discrete, its *probability mass function* is usually indicated by $p(x)$ and its *cumulative distribution function* by $P(x)$. When a random variable's sample space is continuous, its *probability density function* and its *cumulative distribution function* are usually indicated by $f(x)$ and $F(x)$, respectively.

12.3.1 Discrete Random Variables

A random variable X is said to be *discrete* if it can take on at most a countable (finite) number of values $(x_1, x_2, \ldots, x_L)$. The probability that a discrete random variable X takes on the value x_i is given by

$$\Pr\{X = x_i\} = p(x_i) \text{ for } i = 1, 2, \ldots, L \text{ (} i \text{ is a } \textit{sequential index} \text{ of the discrete values, } x_i, \text{ that the variable takes on),}$$

where $p(x_i) \geq 0$ and $\sum_i p(x_i) = 1$.

The probability of events about a discrete random variable can be computed from its probability mass function $p(x)$. For example, the probability of the event that the value of X is contained in the closed interval $[a, b]$ is given by (where the colon is read *such that*)

$$\Pr\{a \leq X \leq b\} = \sum_{i:a \leq X_i \leq b} p(x_i). \tag{12-1}$$

The probability that the value of X is less than or equal to $x = h$, the cumulative distribution function $P(x)$ for a discrete case, is given by

$$\Pr\{X \leq h\} \doteq P(h) = \sum_{i:X_i \leq h} p(x_i). \tag{12-2}$$

In most practical applications, discrete random variables represent *countable* data such as the useful life of an asset in years, number of maintenance jobs per week, or number of employees as positive integers.

12.3.2 Continuous Random Variables

A random variable X is said to be continuous if there exists a nonnegative function $f(x)$ such that, for any set of real numbers $[c, d]$, where $c < d$, the probability of the event that the value of X is contained in the set is given by

$$\Pr\{c \leq X \leq d\} = \int_c^d f(x)\, dx \tag{12-3}$$

and

$$\int_{-\infty}^{\infty} f(x)\, dx = 1.$$

Thus, the probability of events about the continuous random variable X can be computed from its probability density function, and the probability that X assumes exactly any one of its values is zero. Also, the probability that the value of X is less than or equal to a value $x = k$, the cumulative distribution function $F(x)$ for a continuous case, is given by

$$\Pr\{X \leq k\} = F(k) = \int_{-\infty}^{k} f(x)\, dx. \tag{12-4}$$

Also, for a continuous case,

$$\Pr\{c \le X \le d\} = \int_c^d f(x)\,dx = F(d) - F(c). \qquad (12\text{-}5)$$

In most practical applications, continuous random variables represent *measured* data such as time, cost, and revenue on a continuous scale. Depending upon the situation, the analyst decides to model random variables in engineering economic analysis as either discrete or continuous.

12.3.3 Mathematical Expectation and Selected Statistical Moments

The expected value of a single random variable X, E(X), is a weighted average of the distributed values x that it takes on and is a measure of the central location of the distribution (central tendency of the random variable). The E(X) is the first moment of the random variable about the origin and is called the *mean* (central moment) of the distribution. The expected value is

$$E(X) = \left\{ \begin{array}{ll} \sum_i x_i\, p(x_i) & \text{for } x \text{ discrete and } i = 1, 2, \ldots, L \\ \int_{-\infty}^{\infty} x f(x)\,dx & \text{for } x \text{ continuous.} \end{array} \right\} \qquad (12\text{-}6)$$

Although E(X) provides a measure of central tendency, it does not measure how the distributed values x cluster around the mean. The *variance*, V(X), which is nonnegative, of a single random variable X is a measure of the dispersion of the values it takes on around the mean. It is the expected value of the square of the difference between the values x and the mean, which is the second moment of the random variable about its mean:

$$E\{[X - E(X)]^2\} = V(X) = \left\{ \begin{array}{ll} \sum_i [x_i - E(X)]^2\, p(x_i) & \text{for } x \text{ discrete} \\ \int_{-\infty}^{\infty} [x - E(X)]^2 f(x)\,dx & \text{for } x \text{ continuous.} \end{array} \right\} \qquad (12\text{-}7)$$

From the binomial expansion of $[X - E(X)]^2$, it can be easily shown that $V(X) = E(X^2) - [E(X)]^2$. That is, V(X) equals the second moment of the random variable around the origin, which is the expected value of X^2, minus the square of its mean. This is the form often used for calculating the variance of a random variable X:

$$V(X) = \left\{ \begin{array}{ll} \sum_i x_i^2\, p(x_i) - [E(X)]^2 & \text{for } x \text{ discrete} \\ \int_{-\infty}^{\infty} x_i^2 f(x)\,dx - [E(X)]^2 & \text{for } x \text{ continuous.} \end{array} \right\} \qquad (12\text{-}8)$$

The *standard deviation* of a random variable, SD(X), is the positive square root of the variance; that is, $SD(X) = [V(X)]^{1/2}$.

12.3.4 Multiplication of a Random Variable by a Constant

A common operation performed on a random variable is to multiply it by a constant, for example, the estimated maintenance labor expense for a time period, $Y = cX$, when the number of labor hours per period (X) is a random variable, and the cost per labor hour (c) is a constant. Another example is the present worth (PW) calculation for a project when the before-tax or after-tax net cash-flow amounts, F_k, are random variables, and then each F_k is multiplied by a constant $(P/F, i\%, k)$ to obtain the PW value.

When a random variable, X, is multiplied by a constant, c, the expected value, E(cX), and the variance, V(cX), are given by

$$E(cX) = c\,E(X) = \left\{ \begin{array}{ll} \sum_i c x_i\, p(x_i) & \text{for } x \text{ discrete} \\ \int_{-\infty}^{\infty} c x f(x)\,dx & \text{for } x \text{ continuous} \end{array} \right\} \qquad (12\text{-}9)$$

and

$$\begin{aligned} V(cX) &= E\left\{[cX - E(cX)]^2\right\} \\ &= E\left\{c^2X^2 - 2c^2X \cdot E(X) + c^2[E(X)]^2\right\} \\ &= c^2 E\{[X - E(X)]^2\} \\ &= c^2 V(X). \end{aligned} \qquad (12\text{-}10)$$

12.3.5 Multiplication of Two Independent Random Variables

A cash-flow random variable, say Z, may result from the product of two other random variables, $Z = XY$. Sometimes, X and Y can be treated as statistically independent random variables. For example, consider the estimated annual expenses, $Z = XY$, for a repair part repetitively procured during the year on a competitive basis, when the unit price (X) and the number of units used per year (Y) are modeled as independent random variables.

When a random variable, Z, is a product of two independent random variables, X and Y, the expected value, E(Z), and the variance, V(Z), are

$$\begin{aligned} Z &= XY \\ E(Z) &= E(X)\,E(Y); \\ V(Z) &= E[XY - E(XY)]^2 \\ &= E\left\{X^2Y^2 - 2XY\,E(XY) + [E(XY)]^2\right\} \\ &= E(X^2)\,E(Y^2) - [E(X)\,E(Y)]^2. \end{aligned} \qquad (12\text{-}11)$$

But the variance of any random variable, $V(\text{RV})$, is

$$V(\text{RV}) = E[(\text{RV})^2] - [E(\text{RV})]^2,$$
$$E[(\text{RV})^2] = V(\text{RV}) + [E(\text{RV})]^2.$$

Then, $V(Z) = \{V(X) + [E(X)]^2\}\{V(Y) + [E(Y)]^2\} - [E(X)]^2[E(Y)]^2$

or

$$V(Z) = V(X)[E(Y)]^2 + V(Y)[E(X)]^2 + V(X)\,V(Y). \quad (12\text{-}12)$$

12.4 Evaluation of Projects with Discrete Random Variables

Expected value and variance concepts apply theoretically to long-run conditions in which it is assumed that the event is going to occur repeatedly. Application of these concepts, however, is often useful even when investments are not going to be made repeatedly over the long run. In this section, several examples are used to illustrate these concepts with selected economic factors modeled as discrete random variables.

EXAMPLE 12-1 Premixed-Concrete Plant Project

We now apply the expected value and variance concepts to a small premixed-concrete plant project. Suppose that the estimated probabilities of attaining various capacity utilizations are as follows:

Capacity (%)	Probability	Annual Revenue	AW(15%)
50	0.10	$405,000	−$25,093
65	0.30	526,500	22,136
75	0.50	607,500	53,622
90	0.10	729,000	100,850

It is desired to determine the expected value and variance of *annual revenue*. Subsequently, the expected value and variance of annual worth (AW) for the project can be computed. By evaluating both E(AW) and V(AW) for the concrete plant, indications of the venture's average profitability and its riskiness are obtained. The calculations are shown in Tables 12-1 and 12-2.

Solution

Expected value of annual revenue: $\sum(A \times B) = \$575{,}100$.

Variance of annual revenue: $\sum(A \times C) - (575{,}100)^2 = 6{,}360 \times 10^6 (\$)^2$.

Expected value of AW: $\sum(A \times B) = \$41{,}028$

Solution for Annual Revenue (Example 12-1)

i	Capacity (%)	(A) Probability $p(x_i)$	(B) Revenue x_i	(A) × (B) Expected Revenue	(C) = $(B)^2$ x_i^2	(A) × (C)
1	50	0.10	$405,000	$40,500	1.64×10^{11}	0.164×10^{11}
2	65	0.30	526,500	157,950	2.77×10^{11}	0.831×10^{11}
3	75	0.50	607,500	303,750	3.69×10^{11}	1.845×10^{11}
4	90	0.10	729,000	72,900	5.31×10^{11}	0.531×10^{11}
				$575,100		3.371×10^{11} ($)^2

Solution for AW (Example 12-1)

i	Capacity (%)	(A) $p(x_i)$	(B) AW,x_i	(A) × (B) Expected AW	(C) = $(B)^2$ $(\text{AW})^2$	(A) × (C)
1	50	0.10	−$25,093	−$2,509	0.63×10^9	0.063×10^9
2	65	0.30	22,136	6,641	0.49×10^9	0.147×10^9
3	75	0.50	53,622	26,811	2.88×10^9	1.440×10^9
4	90	0.10	100,850	10,085	10.17×10^9	1.017×10^9
				$41,028		2.667×10^9 ($)^2

Variance of AW: $\sum(A \times C) - (41{,}028)^2 = 9{,}837 \times 10^5 (\$)^2$

Standard deviation of AW: $31,364

The standard deviation of AW, SD(AW), is less than the expected AW, E(AW), and only the 50% capacity utilization situation results in a negative AW. Consequently, with this additional information, the investors in this undertaking may well judge the venture to be an acceptable one.

There are projects, such as the flood-control situation in the next example, in which future losses due to natural or human-made risks can be decreased by increasing the amount of capital that is invested. Drainage channels or dams, built to control floodwaters, may be constructed in different sizes, costing different amounts. If they are correctly designed and used, the larger the size, the smaller will be the resulting damage loss when a flood occurs. As we might expect, the most economical size would provide satisfactory protection against most floods, although it could be anticipated that some overloading and damage might occur at infrequent periods.

EXAMPLE 12-2

Channel Enlargement for Flash Flood Control

A drainage channel in a community where flash floods are experienced has a capacity sufficient to carry 700 cubic feet per second. Engineering studies produced the following data regarding the probability that a given water flow in any one year will be exceeded and the cost of enlarging the channel:

Water Flow (ft^3/sec)	Probability of a Greater Flow Occurring in Any One Year	Capital Investment to Enlarge Channel to Carry This Flow
700	0.20	—
1,000	0.10	$20,000
1,300	0.05	30,000
1,600	0.02	44,000
1,900	0.01	60,000

Records indicate that the average property damage amounts to $20,000 when serious overflow occurs. It is believed that this would be the average damage whenever the storm flow is *greater* than the capacity of the channel. Reconstruction of the channel would be financed by 40-year bonds bearing 8% interest per year. It is thus computed that the capital recovery amount for debt repayment (principal of the bond plus interest) would be 8.39% of the capital investment, because $(A/P, 8\%, 40) = 0.0839$. It is desired to determine the most economical channel size (water-flow capacity).

Solution

The total expected equivalent uniform annual cost for the structure and property damage for all alternative channel sizes would be as shown in Table 12-3. These calculations show that the minimum expected annual cost would be achieved by enlarging the channel so that it would carry 1,300 cubic feet per second, with the expectation that a greater flood might occur in 1 year out of 20 on the average and cause property damage of $20,000.

Expected Equivalent Annual Cost (Example 12-2)

Water Flow (ft^3/sec)	Capital Recovery Amount	Expected Annual Property Damage[a]	Total Expected Equivalent Uniform Annual Cost
700	None	$20,000(0.20) = $4,000	$4,000
1,000	$20,000(0.0839) = $1,678	20,000(0.10) = 2,000	3,678
1,300	30,000(0.0839) = 2,517	20,000(0.05) = 1,000	3,517
1,600	44,000(0.0839) = 3,692	20,000(0.02) = 400	4,092
1,900	60,000(0.0839) = 5,034	20,000(0.01) = 200	5,234

[a] These amounts are obtained by multiplying $20,000 by the probability of greater water flow occurring.

Note that when loss of life or limb might result, as in Example 12-2, there usually is considerable pressure to disregard pure economy and build such projects in recognition of the nonmonetary values associated with human safety.

The following example illustrates the same principles as in Example 12-2, except that it applies to safety alternatives involving electrical circuits.

EXAMPLE 12-3

Investing in Circuit Protection to Reduce Probability of Failure

Three alternatives are being evaluated for the protection of electrical circuits, with the following required investments and probabilities of failure:

Alternative	Capital Investment	Probability of Loss in Any Year
A	$90,000	0.40
B	100,000	0.10
C	160,000	0.01

If a loss does occur, it will cost $80,000 with a probability of 0.65 and $120,000 with a probability of 0.35. The probabilities of loss in any year are independent of the probabilities associated with the resultant cost of a loss if one does occur. Each alternative has a useful life of eight years and no estimated market value at that time. The minimum attractive rate of return (MARR) is 12% per year, and annual maintenance expenses are expected to be 10% of the capital investment. It is desired to determine which alternative is best based on expected total annual costs (Table 12-4).

Solution

The expected value of a loss, if it occurs, can be calculated as follows:

$$\$80,000(0.65) + \$120,000(0.35) = \$94,000.$$

Thus, Alternative *B* is the best based on total expected equivalent uniform annual cost, which is a long-run average cost. One might, however, rationally choose Alternative C to reduce significantly the chance of an $80,000 or $120,000 loss occurring in any year in return for a 24.3% increase in the total expected equivalent uniform annual cost.

Expected Equivalent Annual Cost (Example 12-3)

Alternative	Capital Recovery Amount = Capital Investment × (A/P, 12%, 8)	Annual Maintenance Expense = Capital Investment × (0.10)	Expected Annual Cost of Failure	Total Expected Equivalent Annual Cost
A	$90,000(0.2013) = $18,117	$9,000	$94,000(0.40) = $37,600	$64,717
B	100,000(0.2013) = 20,130	10,000	94,000(0.10) = 9,400	39,530
C	160,000(0.2013) = 32,208	16,000	94,000(0.01) = 940	49,148

In Examples 12-1 through 12-3, a revenue or cost factor was modeled as a discrete random variable, with project life assumed certain. A second type of situation involves the cash-flow estimates being certain, but the project life being modeled as a discrete random variable. This is illustrated in Example 12-4.

EXAMPLE 12-4 Project Life as a Random Variable

The heating, ventilating, and air-conditioning (HVAC) system in a commercial building has become unreliable and inefficient. Rental income is being hurt, and the annual expenses of the system continue to increase. Your engineering firm has been hired by the owners to (1) perform a technical analysis of the system, (2) develop a preliminary design for rebuilding the system, and (3) accomplish an engineering economic analysis to assist the owners in making a decision. The estimated capital-investment cost and annual savings in operating and maintenance expenses, based on the preliminary design, are shown in the following table. The estimated annual increase in rental revenue with a modern HVAC system has been developed by the owner's marketing staff and is also provided in the following table. These estimates are considered reliable because of the extensive information available. The useful life of the rebuilt system, however, is quite uncertain. The estimated probabilities of various useful lives are provided. Assume that MARR = 12% per year and the estimated market value of the rebuilt system at the end of its useful life is zero. Based on this information, what are $E(\text{PW})$, $V(\text{PW})$, and SD(PW) of the project's cash flows? Also, what is the probability of PW ≥ 0? What decision would you make regarding the project, and how would you justify your decision using the available information? Solve by hand and by spreadsheet.

Economic Factor	Estimate
Capital investment	$521,000
Annual savings	48,600
Increased annual revenue	31,000

Useful Life, Year (N)	$p(N)$	
12	0.1	
13	0.2	
14	0.3	
15	0.2	$\sum = 1.00$
16	0.1	
17	0.05	
18	0.05	

Solution by Hand

The PW of the project's cash flows, as a function of project life (N), is

$$\text{PW}(12\%)_N = -\$521{,}000 + \$79{,}600(P/A, 12\%, N).$$

The calculation of the value of $E(\text{PW}) = \$9{,}984$ and the value of $E[(\text{PW})^2] = 577.527 \times 10^6(\$)^2$ are shown in Table 12-5. Then, by using Equation (12-8), the

Calculation of $E(\text{PW})$ and $E[(\text{PW})^2]$ (Example 12-4)

(1) Useful Life (N)	(2) PW(N)	(3) $p(N)$	(4) = (2) × (3) E[PW(N)]	(5) = (2)^2 [PW(N)]^2	(6) = (3) × (5) $p(N)$[PW(N)]^2
12	−$27,926	0.1	−$2,793	779.86×10^6	77.986×10^6
13	−9,689	0.2	−1,938	93.88×10^6	18.776×10^6
14	6,605	0.3	1,982	43.63×10^6	13.089×10^6
15	21,148	0.2	4,230	447.24×10^6	89.448×10^6
16	34,130	0.1	3,413	$1{,}164.86 \times 10^6$	116.486×10^6
17	45,720	0.05	2,286	$2{,}090.32 \times 10^6$	104.516×10^6
18	56,076	0.05	2,804	$3{,}144.52 \times 10^6$	157.226×10^6
			E(PW) = $9,984		$E[(\text{PW})^2] = 577.527 \times 10^6(\$)^2$

variance of the PW is

$$V(\text{PW}) = E[(\text{PW})^2] - [E(\text{PW})]^2$$
$$= 577.527 \times 10^6 - (\$9{,}984)^2$$
$$= 477.847 \times 10^6(\$)^2.$$

The SD(PW) is equal to the positive square root of the variance, $V(\text{PW})$:

$$\text{SD(PW)} = [V(\text{PW})]^{1/2} = (477.847 \times 10^6)^{1/2}$$
$$= \$21{,}859.$$

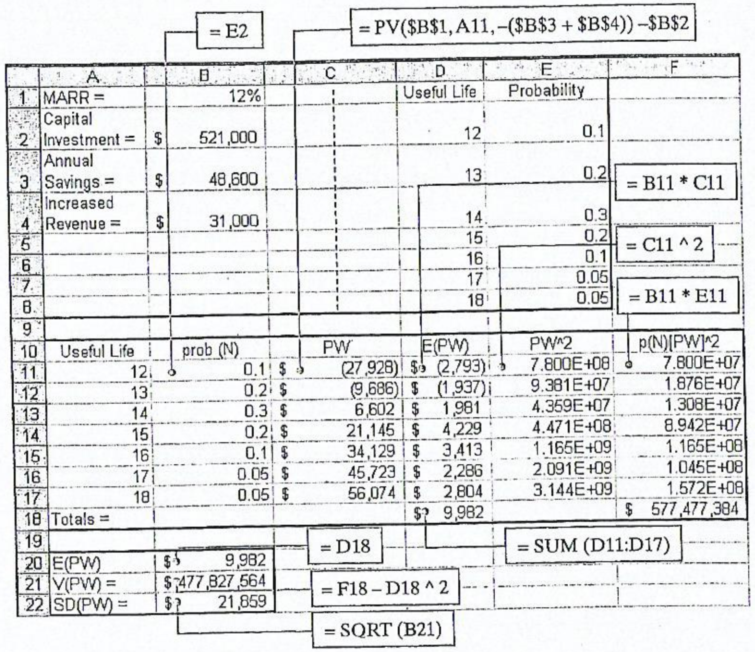
Based on the PW of the project as a function of N (column 2), and the probability of each PW(N) value occurring (column 3), the probability of the PW being ≥ 0 is

$$\Pr\{\text{PW} \geq 0\} = 1 - (0.1 + 0.2) = 0.7.$$

The results of the engineering economic analysis indicate that the project is a questionable business action. The $E(\text{PW})$ of the project is positive ($9,984) but small relative to the large capital investment. Also, even though the probability of the PW being greater than zero is somewhat favorable (0.7), the SD(PW) value is large [over two times the $E(\text{PW})$ value].

Spreadsheet Solution

Spreadsheets are well suited for the number crunching involved in computing expected values and variances associated with discrete probability functions. Figure 12-1 illustrates the use of a spreadsheet to perform the necessary calculations for this example. The basic format of Table 12-5 is used and the results are same, subject to rounding differences. Once the spreadsheet, however, is formulated, we can easily perform sensitivity analyses to support



	A	B	C	D	E	F
1	MARR =	12%		Useful Life	Probability	
2	Capital Investment =	$ 521,000		12	0.1	
3	Annual Savings =	$ 48,600		13	0.2	
4	Increased Revenue =	$ 31,000		14	0.3	
5				15	0.2	
6				16	0.1	
7				17	0.05	
8				18	0.05	
9						
10	Useful Life	prob (N)	PW	E(PW)	PW^2	p(N)[PW]^2
11	12	0.1	$ (27,928)	$ (2,793)	7.800E+08	7.800E+07
12	13	0.2	$ (9,686)	$ (1,937)	9.381E+07	1.876E+07
13	14	0.3	$ 6,602	$ 1,981	4.359E+07	1.308E+07
14	15	0.2	$ 21,145	$ 4,229	4.471E+08	8.942E+07
15	16	0.1	$ 34,129	$ 3,413	1.165E+09	1.165E+08
16	17	0.05	$ 45,723	$ 2,286	2.091E+09	1.045E+08
17	18	0.05	$ 56,074	$ 2,804	3.144E+09	1.572E+08
18	Totals =			$ 9,982		$ 577,477,384
19						
20	E(PW)	$ 9,982				
21	V(PW) =	$ 477,827,564				
22	SD(PW) =	$ 21,859				

Figure 12-1 Spreadsheet Solution, Example 12-4

our recommendation. For example, a 5% decrease in annual savings results in a negative E(PW), supporting our conclusion that the profitability of the project is questionable.

12.4.1 Probability Trees

The discrete distribution of cash flows sometimes occurs in each time period. A *probability tree diagram* is useful in describing the prospective cash flows, and the probability of each value occurring, for this situation. Example 12-5 is a problem of this type.

EXAMPLE 12-5 Project Analysis Using a Probability Tree

The uncertain cash flows for a small improvement project are described by the probability tree diagram in Figure 12-2. (Note that the probabilities emanating from each node sum to unity.) The analysis period is two years, and MARR = 12% per year. Based on this information,

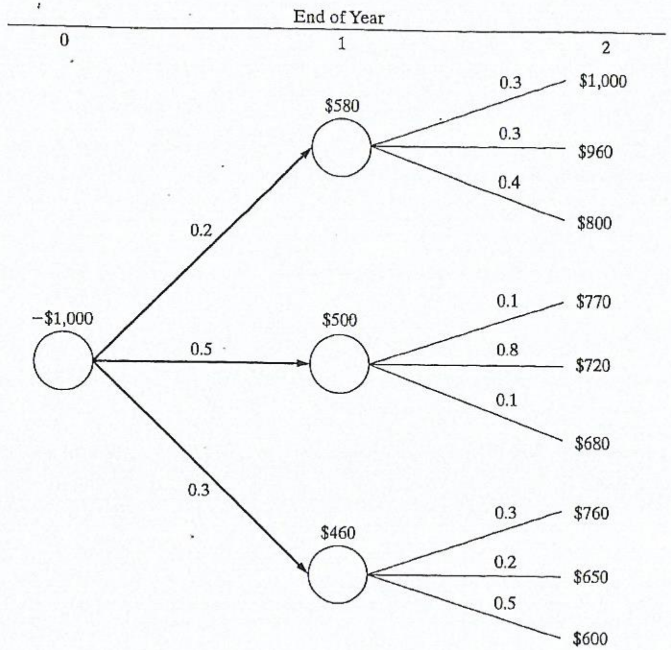


Figure 12-2 Probability Tree Diagram for Example 12-5

(a) What are the E(PW), V(PW), and SD(PW) of the project?

(b) What is the probability that PW ≤ 0?

(c) Which analysis result(s) favor approval of the project, and which ones appear unfavorable?

Solution

(a) The calculation of the values for E(PW) and $E[(\text{PW})^2]$ is shown in Table 12-6. In column 2, PW_j is the PW of branch j in the tree diagram. The probability of each branch occurring, $p(j)$, is shown in column 3. For example, proceeding from the right node for each cash flow in Figure 12-2 to the left node, we have $p(1) = (0.3)(0.2) = 0.06$ and $p(9) = (0.5)(0.3) = 0.15$. Hence,

$$E(\text{PW}) = \sum_j (\text{PW}_j)p(j) = \$39.56.$$

Then, $V(\text{PW}) = E[(\text{PW})^2] - [E(\text{PW})]^2$

$$= 15{,}227 - (\$39.56)^2$$

$$= 13{,}662(\$)^2,$$

Calculation of $E(\text{PW})$ and $E[(\text{PW})^2]$ (Example 12-5)

	(1) Net Cash Flow EOY							
j	0	1	2	(2) PW_j	(3) $p(j)$	(4) = (2) × (3) $E(\text{PW}_j)$	(5) = $(2)^2$ $(\text{PW}_j)^2$	(6) = (3) × (5) $E[(\text{PW}_j)^2]$
1	−$1,000	$580	$1,000	$315	0.06	$18.90	99,225$²	5,953$²
2	−1,000	580	960	283	0.06	16.99	80,089	4,805
3	−1,000	580	800	156	0.08	12.45	24,336	1,947
4	−1,000	500	770	60	0.05	3.04	3,600	180
5	−1,000	500	720	20	0.40	8.17	400	160
6	−1,000	500	680	−11	0.05	−0.57	121	6
7	−1,000	460	760	17	0.09	1.49	289	26
8	−1,000	460	650	−71	0.06	−4.27	5,044	302
9	−1,000	460	600	−111	0.15	−16.64	12,321	1,848
						E(PW) = $39.56		E[(PW)²] = 15,227$²

and $\text{SD(PW)} = [V(\text{PW})]^{1/2} = (13{,}662)^{1/2} = \$116.88$.

(b) Based on the entries in column 2, PW_j, and column 3, $p(j)$, we have

$$\begin{aligned}\Pr\{\text{PW} \le 0\} &= p(6) + p(8) + p(9)\\ &= 0.05 + 0.06 + 0.15\\ &= 0.26.\end{aligned}$$

(c) The analysis results that favor approval of the project are $E(\text{PW}) = \$39.56$, which is greater than zero only by a small amount, and $\Pr\{\text{PW} > 0\} = 1 - 0.26 = 0.74$. The SD(PW) = $116.92, however, is approximately three times the $E(\text{PW})$. This indicates a relatively high variability in the measure of economic merit, PW of the project, and is usually an unfavorable indicator of project acceptability.

12.4.2 An Application Perspective

One of the major problems in computing expected values is the determination of the probabilities. In many situations, there is no precedent for the particular venture being considered. Therefore, probabilities seldom can be based on historical data and rigorous statistical procedures. In most cases, the analyst, or the person making the decision, must make a judgment based on all available information in estimating the probabilities. This fact makes some people hesitate to use the expected-value concept, because they cannot see the value of applying such a technique to improve the evaluation of risk and uncertainty when so much apparent subjectivity is present.

Although this argument has merit, the fact is that engineering economy studies deal with future events and there must be an extensive amount of estimating. Furthermore, even if the probabilities could be based accurately on past history, there rarely is any assurance that the future will repeat the past. Thus, structured methods for assessing subjective probabilities are often used in practice.* Also, even if we must estimate the probabilities, the very process of doing so requires us to make explicit the uncertainty that is inherent in all estimates going into the analysis. Such structured thinking is likely to produce better results than little or no thinking about such matters.

12.5 Evaluation of Projects with Continuous Random Variables

In this section, we continue to compute the expected values and the variances of probabilistic factors, but we model the selected probabilistic factors as *continuous* random variables. In each example, simplifying assumptions are made about the distribution of the random variable and the statistical relationship among the values it takes on. When the situation is more complicated, such as a problem that involves probabilistic cash flows and probabilistic project lives, a second general procedure that utilizes Monte Carlo simulation is normally used. This is the subject of Section 12.6.

Two frequently used assumptions about uncertain cash-flow amounts are that they are distributed according to a normal distribution† and are statistically independent. Underlying these assumptions is a general characteristic of many cash flows, in that they result from a number of different and independent factors.

The advantage of using statistical independence as a simplifying assumption, when appropriate, is that no correlation between the cash-flow amounts (e.g., the net annual cash-flow amounts for an alternative) is being assumed. Consequently, if we have a linear combination of two or more independent cash-flow amounts, say $\text{PW} = c_0F_0 + \cdots + c_NF_N$, where the c_k values are coefficients and the F_k values are periodic net cash flows, the expression for the $V(\text{PW})$, based on Equation (12-10), reduces to

$$V(\text{PW}) = \sum_{k=0}^{N} c_k^2 V(F_k). \tag{12-13}$$

* For further information, see W. G. Sullivan and W. W. Claycombe, *Fundamentals of Forecasting* (Reston, VA: Reston Publishing Co., 1977), Chapter 6.

† This frequently encountered continuous probability function is discussed in any good statistics book, such as R. E. Walpole and R. H. Myers, *Probability and Statistics for Engineers and Scientists* (New York: Macmillan Publishing Co., 1989), 139–154.

And, based on Equation (12-9), we have

$$E(\text{PW}) = \sum_{k=0}^{N} c_k E(F_k). \qquad (12\text{-}14)$$

EXAMPLE 12-6 Annual Net Cash Flow as a Continuous Random Variable

For the following annual cash-flow estimates, find the $E(\text{PW})$, $V(\text{PW})$, and SD(PW) of the project. Assume that the annual net cash-flow amounts are normally distributed with the expected values and standard deviations as given and statistically independent and that the MARR = 15% per year.

End of Year, k	Expected Value of Net Cash Flow, F_k	SD of Net Cash Flow, F_k
0	−$7,000	0
1	3,500	$600
2	3,000	500
3	2,800	400

A graphical portrayal of these normally distributed cash flows is shown in Figure 12-3.

Solution

The expected PW, based on Equation (12-14), is calculated as follows, where $E(F_k)$ is the expected net cash flow in year k $(0 \le k \le N)$ and c_k is the single-payment

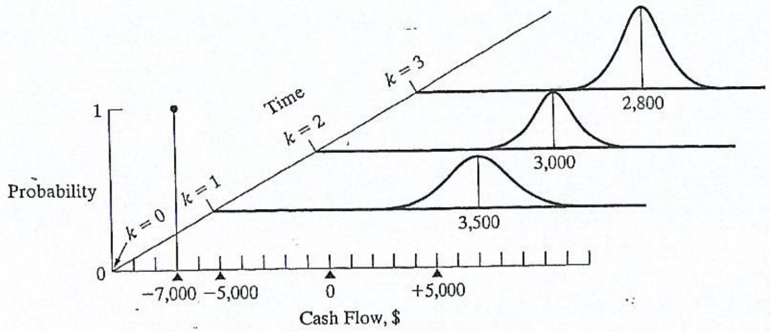


Figure 12-3 Probabilistic Cash Flows over Time (Example 12-6)

PW factor $(P/F, 15\%, k)$:

$$\begin{aligned}E(\text{PW}) &= \sum_{k=0}^{3} (P/F,\ 15\%,\ k)E(F_k)\\ &= -\$7{,}000 + \$3{,}500(P/F,\ 15\%,\ 1) + \$3{,}000(P/F,\ 15\%,\ 2)\\ &\quad + \$2{,}800(P/F,\ 15\%,\ 3)\\ &= \$153.\end{aligned}$$

To determine $V(\text{PW})$, we use the relationship in Equation (12-13). Thus,

$$\begin{aligned}V(\text{PW}) &= \sum_{k=0}^{3} (P/F,\ 15\%,\ k)^2 V(F_k)\\ &= 0^2 1^2 + 600^2(P/F,\ 15\%,\ 1)^2 + 500^2(P/F,\ 15\%,\ 2)^2\\ &\quad + 400^2(P/F,\ 15\%,\ 3)^2\\ &= 484{,}324\$^2\end{aligned}$$

and

$$\text{SD(PW)} = [V(\text{PW})]^{1/2} = \$696.$$

When we can assume that a random variable, say the PW of a project's cash flow, is normally distributed with a mean, $E(\text{PW})$, and a variance, $V(\text{PW})$, we can compute the probability of events about the random variable occurring. This assumption can be made, for example, when we have some knowledge of the shape of the distribution of the random variable and when it is appropriate to do so. Also, this assumption may be supportable when a random variable, such as a project's PW value, is a linear combination of other independent random variables (say, the cash-flow amounts, F_k), regardless of whether the form of the probability distribution(s) of these variables is known.*

EXAMPLE 12-7 Probability of an Unfavorable Project Outcome

Refer to Example 12-6. For this problem, what is the probability that the internal rate of return (IRR) of the cash-flow estimates is less than MARR, Pr{IRR < MARR}? Assume that the PW of the project is a normally distributed random variable, with its mean and variance equal to the values calculated in Example 12-6.

* The theoretical basis of this assumption is the central limit theorem of statistics. For a summary discussion of the supportability of this assumption under different conditions, see C. S. Park and G. P. Sharpe-Bette, *Advanced Engineering Economics* (New York: John Wiley & Sons, 1990), 420–421.

Solution

For a decreasing PW(i) function having a unique IRR, the probability that IRR is less than MARR is the same as the probability that PW is less than zero. Consequently, by using the standard normal distribution in Appendix E, we can determine the probability that PW is less than zero:*

$$Z = \frac{PW - E(PW)}{SD(PW)} = \frac{0 - 153}{696} = -0.22;$$

$$\Pr\{PW \leq 0\} = \Pr\{Z \leq -0.22\}.$$

From Appendix E, we find that $\Pr\{Z \leq -0.22\} = 0.4129$.

EXAMPLE 12-8

Linear Combination of Independent Random Variables

The estimated cash-flow data for a project are shown in the following table for the five-year study period being used. Each annual net cash-flow amount, F_k, is a linear combination of two statistically independent random variables, X_k and Y_k, where X_k is a revenue factor and Y_k is an expense factor. The X_k cash-flow amounts are statistically independent of each other, and the same is true of the Y_k amounts. Both X_k and Y_k are continuous random variables, but the form of their probability distributions is not known. MARR = 20% per year. Based on this information,

(a) What are the E(PW), V(PW), and SD(PW) values of the project's cash flows?
(b) What is the probability that PW is less than zero, that is, $\Pr\{PW \leq 0\}$, and the project is economically attractive?

End of Year, k	Net Cash Flow $F_k = a_kX_k - b_kY_k$	Expected Value X_k	Expected Value Y_k	Standard Deviation (SD) X_k	Standard Deviation (SD) Y_k
0	$F_0 = X_0 + Y_0$	$0	−$100,000	$0	$10,000
1	$F_1 = X_1 + Y_1$	60,000	−20,000	4,500	2,000
2	$F_2 = X_2 + 2Y_2$	65,000	−15,000	8,000	1,200
3	$F_3 = 2X_3 + 3Y_3$	40,000	−9,000	3,000	1,000
4	$F_4 = X_4 + 2Y_4$	70,000	−20,000	4,000	2,000
5	$F_5 = 2X_5 + 2Y_5$	55,000	−18,000	4,000	2,300

* The standard normal distribution, $f(Z)$, of the variable $Z = (X - \mu)/\sigma$ has a mean of zero and a variance of one, when X is a normally distributed random variable with mean μ and standard deviation σ.

Calculation of $E(F_k)$ and $V(F_k)$ (Example 12-8)

End of Year, k	F_k	$E(F_k) = a_kE(X_k) + b_kE(Y_k)$	$V(F_k) = a_k^2V(X_k) + b_k^2V(Y_k)$
0	F_0	\$0 − \$100,000 = −\$100,000	$0 + (1)^2(10{,}000)^2 = 100.0 \times 10^6\$^2$
1	F_1	60,000 − 20,000 = 40,000	$(4{,}500)^2 + (1)^2\,(2{,}000)^2 = 24.25 \times 10^6$
2	F_2	65,000 − 2(15,000) = 35,000	$(8{,}000)^2 + (2)^2\,(1{,}200)^2 = 69.76 \times 10^6$
3	F_3	2(40,000) − 3(9,000) = 53,000	$(2)^2(3{,}000)^2 + (3)^2\,(1{,}000)^2 = 45.0 \times 10^6$
4	F_4	70,000 − 2(20,000) = 30,000	$(4{,}000)^2 + (2)^2\,(2{,}000)^2 = 32.0 \times 10^6$
5	F_5	2(55,000) − 2(18,000) = 74,000	$(2)^2(4{,}000)^2 + (2)^2\,(2{,}300)^2 = 85.16 \times 10^6$

Solution

(a) The calculations of the $E(F_k)$ and $V(F_k)$ values of the project's annual net cash flows are shown in Table 12-7. The E(PW) is calculated by using Equation (12-14) as follows:

$$\begin{aligned} E(PW) &= \sum_{k=0}^{5} (P/F, 20\%, k)E(F_k) \\ &= -\$100{,}000 + \$40{,}000(P/F, 20\%, 1) \\ &\quad + \cdots + \$74{,}000(P/F, 20\%, 5) \\ &= \$32{,}517. \end{aligned}$$

Then, the V(PW) is calculated using Equation (12-13) as follows:

$$\begin{aligned} V(PW) &= \sum_{k=0}^{5} (P/F, 20\%, k)^2V(F_k) \\ &= 100.0 \times 10^6 + (24.25 \times 10^6)(P/F, 20\%, 1)^2 \\ &\quad + \cdots + (85.16 \times 10^6)(P/F, 20\%, 5)^2 \\ &= 186.75 \times 10^6(\$)^2. \end{aligned}$$

Finally, $$\begin{aligned} SD(PW) &= [V(PW)]^{1/2} \\ &= [186.75 \times 10^6]^{1/2} \\ &\doteq \$13{,}666. \end{aligned}$$

(b) The PW of the project's net cash flow is a linear combination of the annual net cash-flow amounts, F_k, that are independent random variables. Each of these random variables, in turn, is a linear combination of the independent random variables X_k and Y_k. We can also observe in Table 12-7 that the V(PW) calculation does not include any dominant $V(F_k)$ values. Therefore, we have a reasonable basis on which to assume that the PW of the project's net cash flow is approximately normally distributed, with $E(\text{PW}) = \$32{,}517$ and $\text{SD(PW)} = \$13{,}666$.

Based on this assumption, we have

$$Z = \frac{\text{PW} - E(\text{PW})}{\text{SD(PW)}} = \frac{0 - \$32{,}517}{\$13{,}666} = -2.3794;$$

$$\Pr\{\text{PW} \leq 0\} = \Pr\{Z \leq -2.3794\}.$$

From Appendix E, we find that $\Pr\{Z \leq -2.3794\} = 0.0087$. Therefore, the probability of loss on this project is negligible. Based on this result, the $E(\text{PW}) > 0$, and the $\text{SD(PW)} = 0.42[E(\text{PW})]$; therefore, the project is economically attractive and has low risk of failing to add value to the firm.

12.6 Evaluation of Risk and Uncertainty by Monte Carlo Simulation

The modern development of computers and related software has resulted in the increased use of Monte Carlo simulation as an important tool for analysis of project uncertainties. For complicated problems, Monte Carlo simulation generates random outcomes for probabilistic factors so as to imitate the randomness inherent in the original problem. In this manner, a solution to a rather complex problem can be inferred from the behavior of these random outcomes.

To perform a simulation analysis, the first step is to construct an analytical model that represents the actual decision situation. This may be as simple as developing an equation for the PW of a proposed industrial robot in an assembly line or as complex as examining the economic effects of proposed environmental regulations on typical petroleum refinery operations. The second step is to develop a probability distribution from subjective or historical data for each uncertain factor in the model. Sample outcomes are randomly generated by using the probability distribution for each uncertain quantity and are then used to determine a *trial* outcome for the model. Repeating this sampling process a large number of times leads to a frequency distribution of trial outcomes for a desired measure of merit, such as PW or AW. The resulting frequency distribution can then be used to make probabilistic statements about the original problem.

This section has been adapted from W. G. Sullivan and R. Gordon Orr, "Monte Carlo Simulation Analyzes Alternatives in Uncertain Economy," *Industrial Engineering*, 14, no. 11 (November 1982): 42–49. Reprinted with permission from the Institute of Industrial Engineers, 3577 Parkway Lane, Suite 200, Norcross, GA, 30092, www.iienet.org.

To illustrate the Monte Carlo simulation procedure, suppose that the probability distribution for the useful life of a piece of machinery has been estimated as shown in Table 12-8. The useful life can be simulated by assigning random numbers to each value such that they are proportional to the respective probabilities. (A random number is selected in a manner such that each number has an equal probability of occurrence.) Because two-digit probabilities are given in Table 12-8, random numbers can be assigned to each outcome, as shown in Table 12-9. Next, a single outcome is simulated by choosing a number at random from a table of random numbers.* For example, if any random number between and including 00 and 19 is selected, the useful life is three years. As a further example, the random number 74 corresponds to a life of seven years.

If the probability distribution that describes a random variable is *normal*, a slightly different approach is followed. Here the simulated outcome is based on the mean and standard deviation of the probability distribution and on a random normal deviate, which is a random number of standard deviations above or below the mean of a standard normal distribution. An abbreviated listing of typical random normal deviates is shown in Table 12-10. For normally distributed random variables, the simulated outcome is based on Equation (12-15):

$$\text{Outcome value} = \text{mean} + [\text{random normal deviate} \times \text{standard deviation}]. \quad (12\text{-}15)$$

For example, suppose that an *annual* net cash flow is assumed to be normally distributed, with a mean of \$50,000 and a standard deviation of \$10,000, as shown in Figure 12-4.

Probability Distribution for Useful Life

Number of Years, N		$p(N)$	
3	possible values	0.20	$\sum p(N) = 1.00$
5		0.40	
7		0.25	
10		0.15	

Assignment of Random Numbers

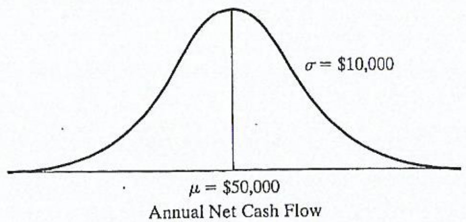
Number of Years, N	Random Numbers
3	00–19
5	20–59
7	60–84
10	85–99

* The last two digits of randomly chosen telephone numbers in a telephone directory are usually quite close to being random numbers.

Random Normal Deviates (RNDs)

−1.565	0.690	−1.724	0.705	0.090
0.062	−0.072	0.778	−1.431	0.240
0.183	−1.012	−0.844	−0.227	−0.448
−0.506	2.105	0.983	0.008	0.295
1.613	−0.225	0.111	−0.642	−0.292

Figure 12-4 Normally Distributed Annual Cash Flow



Example of the Use of RNDs

Year	RND	Annual Net Cash Flow [$50,000 + RND ($10,000)]
1	0.090	$50,900
2	0.240	52,400
3	−0.448	45,520
4	0.295	52,950
5	−0.292	47,080

Simulated cash flows for a period of five years are listed in Table 12-11. Notice that the average annual net cash flow is $248,850/5, which equals $49,770. This approximates the known mean of $50,000 with an error of 0.46%.

If the probability distribution that describes a random event is *uniform* and continuous, with a minimum value of A and a maximum value of B, another procedure should be followed to determine the simulated outcome. Here the outcome can be computed with the formula

$$\text{Simulation outcome} = A + \frac{\text{RN}}{\text{RN}_m}[B - A], \qquad (12\text{-}16)$$

where RN_m is the maximum possible random number (9 if one digit is used, 99 if two are used, etc.) and RN is the random number actually selected. This equation should be used when the minimum outcome, A, and the maximum outcome, B, are known.

For example, suppose that the market value in year N is assumed to be uniformly and continuously distributed between $8,000 and $12,000. A value of this random variable would be generated as follows with a random number of 74:

$$\text{Simulation outcome} = \$8{,}000 + \frac{74}{99}(\$12{,}000 - \$8{,}000) = \$10{,}990.$$

Proper use of these procedures, coupled with an accurate model, will result in an approximation of the actual outcome. But how many simulation trials are necessary for an *accurate* approximation of, for example, the average outcome? In general, the greater the number of trials, the more accurate the approximation of the mean and standard deviation will be. One method of determining whether a sufficient number of trials has been conducted is to keep a running average of results. At first, this average will vary considerably from trial to trial. The amount of change between successive averages should decrease as the number of simulation trials increases. Eventually, this running (cumulative) average should level off at an accurate approximation.

EXAMPLE 12-9

Project Analysis Using Monte Carlo Simulation

Monte Carlo simulation can also simplify the analysis of a more complex problem. The following estimates relate to an engineering project being considered by a large manufacturer of air-conditioning equipment. Subjective probability functions have been estimated for the four independent uncertain factors as follows:

(a) Capital investment: Normally distributed with a mean of $50,000 and a standard deviation of $1,000.

(b) Useful life: Uniformly and continuously distributed with a minimum life of 10 years and a maximum life of 14 years.

(c) Annual revenue:

$35,000 with a probability of 0.4
$40,000 with a probability of 0.5
$45,000 with a probability of 0.1

(d) Annual expense: Normally distributed, with a mean of $30,000 and a standard deviation of $2,000.

The management of this company wishes to determine whether the capital investment in the project will be a profitable one. The interest rate is 10% per year. To answer this question, the PW of the venture will be simulated.

Solution

To illustrate the Monte Carlo simulation procedure, five trial outcomes are computed manually in Table 12-12. The estimate of the average PW based on this very small sample is $19,010/5 = $3,802. For more accurate results, hundreds or even thousands of repetitions would be required.

Monte Carlo Simulation of PW Involving Four Independent Factors (Example 12-9)

Trial Number	Random Normal Deviate (RND_1)	Capital Investment, I [\$50,000 + RND_1(\$1,000)]	Three-Digit RNs	Project Life, N $[10 + \frac{RN}{999}(14 - 10)]$	Project Life, N (Nearest Integer)
1	−1.003	\$48,997	807	13.23	13
2	−0.358	49,642	657	12.63	13
3	+1.294	51,294	488	11.95	12
4	−0.019	49,981	282	11.13	11
5	+0.147	50,147	504	12.02	12

	One-Digit RN	Annual Revenue, R \$35,000 for 0–3 40,000 for 4–8 45,000 for 9	RND_2	Annual Expense, E [\$30,000 + RND_2 (\$2,000)]	$PW = -I + (R - E)(P/A, 10\%, N)$
1	2	\$35,000	−0.036	\$29,928	−\$12,969
2	0	35,000	+0.605	31,210	−22,720
3	4	40,000	+1.470	32,940	−3,189
4	9	45,000	+1.864	33,728	+23,232
5	8	40,000	−1.223	27,554	+34,656
				Total	+\$19,010

The applications of Monte Carlo simulation for investigating risk and uncertainty are many and varied. Remember that the results, however, can be no more accurate than the model and the probability estimates used. In all cases, the procedure and rules are the same: careful study of the problem and development of the model; accurate assessment of the probabilities involved; true randomization of outcomes as required by the Monte Carlo simulation procedure; and calculation and analysis of the results. Furthermore, a sufficiently large number of Monte Carlo trials should always be used to reduce the estimation error to an acceptable level.

12.7 Performing Monte Carlo Simulation with a Computer

It is apparent from the preceding section that a Monte Carlo simulation of a complex project requiring several thousand trials can be accomplished only with the help of a computer. Indeed, there are numerous simulation programs that can be obtained from software companies and universities. MS Excel also has a simulation feature that can generate random data for seven different probability distributions. In this section, we will demonstrate that Monte Carlo simulation is not only feasible but also relatively easy to perform with spreadsheets.

At the heart of Monte Carlo simulation is the generation of random numbers. Most spreadsheet packages include a RAND( ) function that returns a random number between zero and one. Other advanced statistical functions, such as NORMSINV( ), will return the inverse of a cumulative distribution function (the standard normal distribution in this case). This function can be used to generate random normal deviates. The spreadsheet model shown in Figure 12-5 makes use of these functions in performing a Monte Carlo simulation for the project being evaluated in Example 12-9.

The probabilistic functions for the four independent uncertain factors are identified in the spreadsheet model. The input parameter names and distributions

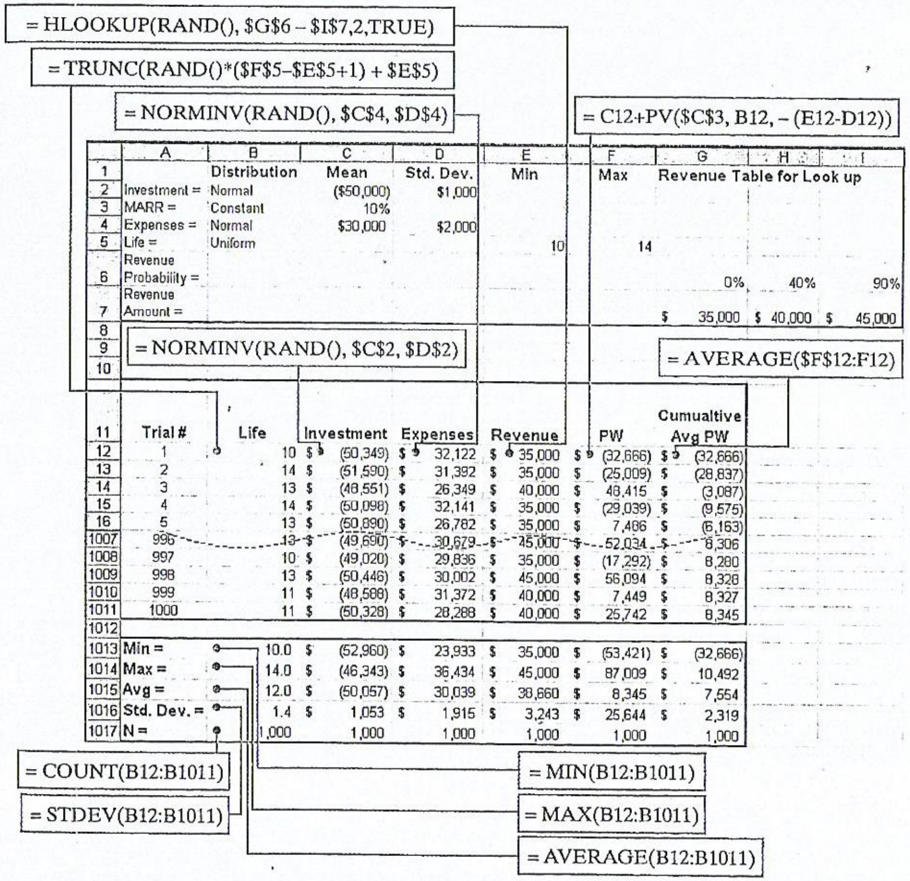


Figure 12-5 Monte Carlo Simulation Using a Spreadsheet, Example 12-9

are identified in cells A2:B7. The mean and standard deviation of the normally distributed factors (capital investment and annual expenses) are identified in the range C2:D4. The maximum and minimum values for the uniformly distributed project life are shown in E5:F5. A discrete probability function was compiled for annual revenues. The associated cumulative probability distribution is shown in G6:I7.

The formulas in cells B12:E12 return values for the uncertain factors, based on the underlying distribution and parameters. These values are used in cell F12 to calculate the PW. As these formulas are copied down the spreadsheet, each row becomes a new trial in the simulation. Each time the F9 (recalculate) key is pressed, the RAND( ) functions are reevaluated and a new simulation begins. The formulas in the range B1013:F1017 calculate basic statistics for the input parameters and the PW. By comparing these results with the input values, one can rapidly check for errors in the basic formulas.

A true simulation involves hundreds, or perhaps thousands, of trials. In this example, 1,000 trials were used. The first and last five trials are shown in Figure 12-5 (rows 17 through 1006 have been hidden for display purposes). The average PW is $8,345 (cell F1015), which is larger than the $3,802 obtained from Table 12-12. This underscores the importance of having a sufficient number of simulation trials to ensure reasonable accuracy in Monte Carlo analyses. The cumulative average PW calculated in column G is one simple way to determine whether enough trials have been run. A plot of these values will stabilize and show little change when the simulation has reached a steady state. Figure 12-6 shows the plot of the cumulative average PW for the 1,000 trials generated for this example.

The average PW is of interest, but the distribution of PW is often even more significant. A histogram of the PW values in column F displays the shape of the distribution, as shown in Figure 12-7. The dispersion of PW trial outcomes in this

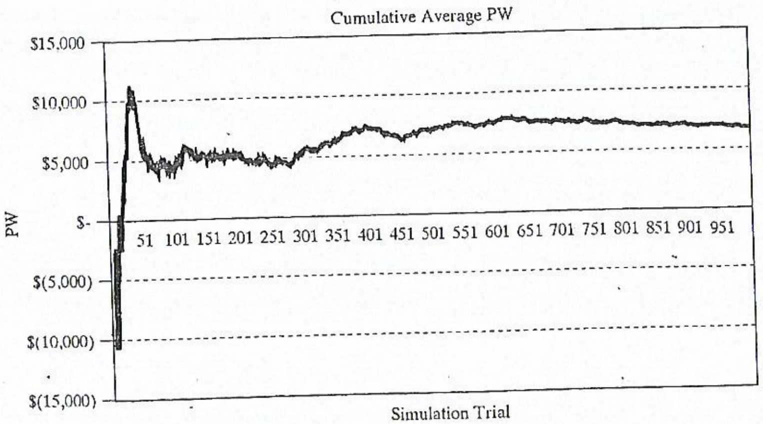


Figure 12-6 Plot of Cumulative Average PW to Determine whether Enough Simulation Trials Have Been Run

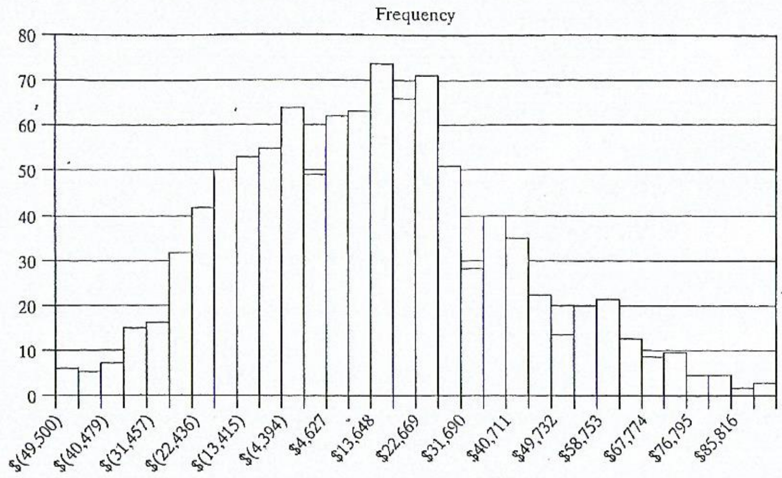


Figure 12-7 Histogram Showing Distribution of PW for 1,000 Simulation Trials

example is considerable. The standard deviation of simulated trial outcomes is one way to measure this dispersion. In addition, the proportion of values less than zero indicates the probability of a negative PW. Based on Figure 12-7, approximately 40% of all simulation outcomes have a PW less than zero. Consequently, this project may be too risky for the company to undertake, because the downside risk of failing to realize at least a 10% per year return on the capital investment is about 4 chances out of 10. Perhaps another project investment should be considered.

A typical application of simulation involves the analysis of several mutually exclusive alternatives. In such studies, how can one compare alternatives that have different expected values and standard deviations of, for instance, PW? One approach is to select the alternative that *minimizes* the probability of attaining a PW that is less than zero. Another popular response to this question utilizes a graph of expected value (a measure of the reward) plotted against standard deviation (an indicator of risk) for each alternative. An attempt is then made to assess subjectively the trade-offs that result from choosing one alternative over another in pairwise comparisons.

To illustrate the latter concept, suppose that three alternatives having varying degrees of uncertainty have been analyzed with Monte Carlo computer simulation and the results shown in Table 12-13 have been obtained. These results are plotted in Figure 12-8, where it is apparent that Alternative C is inferior to Alternatives *A* and *B* because of its lower *E*(PW) and larger standard deviation. Therefore, C offers a smaller PW that has a greater amount of risk associated with it! Unfortunately, the choice of *B* versus *A* is not as clear because the increased expected PW of *B* has to be balanced against the increased risk of *B*. This trade-off *may or may not* favor *B*, depending on management's attitude toward accepting the additional uncertainty

Simulation Results for Three Mutually Exclusive Alternatives

Alternative	E(PW)	SD(PW)	E(PW) ÷ SD(PW)
A	$37,382	$1,999	18.70
B	49,117	2,842	17.28
C	21,816	4,784	4.56

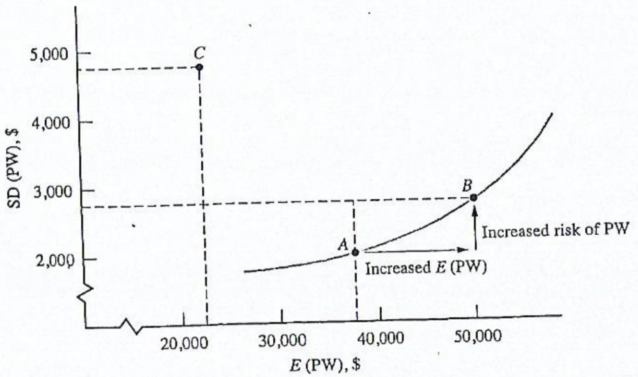


Figure 12-8 Graphical Summary of Computer Simulation Results

associated with a larger expected reward. The comparison also presumes that Alternative *A* is acceptable to the decision maker. One simple procedure for choosing between *A* and *B* is to rank alternatives based on the ratio of E(PW) to SD(PW). In this case, Alternative *A* would be chosen because it has the more favorable (larger) ratio.

12.8 Decision Trees

Decision trees, also called *decision flow networks* and *decision diagrams*, are powerful means of depicting and facilitating the analysis of important problems, especially those that involve sequential decisions and variable outcomes over time. Decision trees are used in practice because they make it possible to break down a large, complicated problem into a series of smaller simple problems, and they enable objective analysis and decision making that includes explicit consideration of the risk and effect of the future.

This section has been adapted (except Section 12.8.3) from John R. Canada and William G. Sullivan, *Economic and Multiattribute Evaluation of Advanced Manufacturing Systems* (Upper Saddle River, NJ: Prentice Hall, 1989), 341–343, 347. Reprinted by permission of the publisher.

The name *decision tree* is appropriate, because it shows branches for each possible alternative for a given decision and for each possible outcome (event) that can result from each alternative. Such networks reduce abstract thinking to a logical visual pattern of cause and effect. When costs and benefits are associated with each branch and probabilities are estimated for each possible outcome, analysis of the decision flow network can clarify choices and risks.

12.8.1 Deterministic Example

The most basic form of a decision tree occurs when each alternative can be assumed to result in a single outcome—that is, when certainty is assumed. The replacement problem in Figure 12-9 illustrates this. The problem illustrates that the decision on whether to replace the defender (old machine) with the new machine (challenger) is not just a one-time decision but rather one that recurs periodically. That is, if the decision is made to keep the old machine at decision point one, then later, at decision point two, a choice again has to be made. Similarly, if the old machine is chosen at decision point two, then a choice again has to be made at decision point three. For each alternative, the cash inflow and duration of the project are shown above the arrow, and the capital investment is shown below the arrow.

For this problem, the question initially seems to be which alternative to choose at decision point 1. But an intelligent choice at decision point 1 should take into account the later alternatives and decisions that stem from it. Hence, the correct procedure is to start at the most distant decision point, determine the best alternative and quantitative result of that alternative, and then roll back to each preceding decision point, repeating the procedure until finally the choice at the initial or present decision point is determined. By this procedure, one can make a present decision that directly takes into account the alternatives and expected decisions of the future.

For simplicity in this example, timing of the monetary outcomes will first be neglected, which means that a dollar has the same value regardless of the year in which it occurs. Table 12-14 shows the necessary computations and decisions using

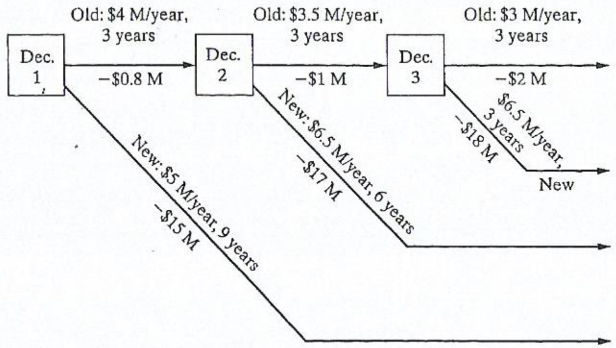


Figure 12-9 Deterministic Replacement Example

Monetary Outcomes and Decisions at Each Point—Deterministic Replacement Example of Figure 12-9[a]

Decision Point	Alternative	Monetary Outcome		Choice
3	Old	\$3M(3) − \$2M	= \$7.0M	Old
	New	\$6.5M(3) − \$18M	= \$1.5M	
2	Old	\$7M + \$3.5M(3) − \$1M	= \$16.5M	
	New	\$6.5M(6) − \$17M	= \$22.0M	New
1	Old	\$22.0M + \$4M(3) − \$0.8M	= \$33.2M	Old
	New	\$5M(9) − \$15M	= \$30.0M	

[a] Interest = 0% per year, that is, ignore timing of cash flows.

a nine-year study period. Note that the monetary outcome of the best alternative at decision point 3 (\$7.0M for the *old*) becomes part of the outcome for the old alternative at decision point 2. Similarly, the best alternative at decision point 2 (\$22.0M for the *new*) becomes part of the outcome for the defender alternative at decision point 1.

The computations in Table 12-14 show that the answer is to keep the old alternative now and plan to replace it with the new one at the end of three years (at decision point 2). But this does not mean that the old machine should necessarily be kept for a full three years and a new machine bought without question at the end of that period. Conditions may change at any time, necessitating a fresh analysis—probably a decision tree analysis—based on estimates that are reasonable in light of conditions at that time.

12.8.1.1 Deterministic Example Considering Timing For decision tree analyses, which involve working from the most distant to the nearest decision point, the easiest way to take into account the timing of money is to use the PW approach and thus *discount all monetary outcomes to the decision points in question*. To demonstrate, Table 12-15 shows computations for the same replacement problem of Figure 12-9, using an interest rate of 25% per year.

Note from Table 12-15 that, when taking into account the effect of timing by calculating PWs at each decision point, the indicated choice is not only to keep the old alternative at decision point 1 but also to keep the old alternative at decision points 2 and 3 as well. This result is not surprising since the high interest rate tends to favor the alternatives with lower capital investments, and it also tends to place less weight on long-term returns (benefits).

12.8.2 General Principles of Diagramming

The proper diagramming of a decision problem is, in itself, generally very useful to the understanding of the problem, and it is essential to correct subsequent analysis.

The placement of decision points (nodes) and chance outcome nodes from the initial decision point to the base of any later decision point should give an accurate representation of the information that will and will not be available when the choice

Decision at Each Point with Interest = 25% per Year for Deterministic Replacement Example of Figure 12-9

Decision Point	Alternative	PW of Monetary Outcome		Choice
3	Old	\$3M(P/A,3) − \$2M \$3M(1.95) − \$2M	= \$3.85M	Old
	New	\$6.5M(P/A,3) − \$18M \$6.5M(1.95) − \$18M	= −\$5.33M	
2	Old	\$3.85M(P/F,3) + \$3.5M(P/A,3) − \$1M \$3.85M(0.512) + \$3.5M(1.95) − \$1M	= \$7.79M	Old
	New	\$6.5M(P/A,6) − \$17M \$6.5M(2.95) − \$17M	= \$2.18M	
1	Old	\$7.79M(P/F,3) + \$4M(P/A,3) − \$0.8M \$7.79M(0.512) + \$4M(1.95) − \$0.8M	= \$10.99M	Old
	New	\$5.0M(P/A,9) − \$15M \$5.0M(3.46) − \$15M	= \$2.30M	

represented by the decision point in question actually has to be made. The decision tree diagram should show the following (normally, a square symbol is used to depict a decision node, and a circle symbol is used to depict a chance outcome node):

1. all initial or immediate alternatives among which the decision maker wishes to choose
2. all uncertain outcomes and future alternatives that the decision maker wishes to consider because they may directly affect the consequences of initial alternatives
3. all uncertain outcomes that the decision maker wishes to consider because they may provide information that can affect his or her future choices among alternatives and hence indirectly affect the consequences of initial alternatives

Note that the alternatives at any decision point and the outcomes at any chance outcome node must be

1. mutually exclusive (i.e., no more than one can be chosen)
2. collectively exhaustive (i.e., one event must be chosen or something must occur if the decision point or outcome node is reached)

12.8.3 Decision Trees with Random Outcomes

The deterministic replacement problem discussed in Section 12.8.1 introduced the concept of sequential decisions using assumed certainty for alternative outcomes. An engineering problem requiring sequential decisions, however, often includes random outcomes, and decision trees are very useful in structuring this type of situation. The decision tree diagram helps to make the problem explicit and assists in its analysis. This is illustrated in Example 12-10.

EXAMPLE 12-10 Probabilistic Decision Tree Analysis

The Ajax Corporation manufactures compressors for commercial air-conditioning systems. A new compressor design is being evaluated as a potential replacement for the most frequently used unit. The new design involves major changes that have the expected advantage of better operating efficiency. From the perspective of a typical user, the new compressor (as an assembled component in an air-conditioning system) would have an increased investment of \$8,600 relative to the present unit and an annual expense saving dependent upon the extent to which the design goal is met in actual operations.

Estimates by the multidisciplinary design team of the new compressor achieving four levels (percentages) of the efficiency design goal and the probability and annual expense saving at each level are as follows:

Level (Percentage) of Design Goal Met (%)	Probability $p(L)$	Annual Expense Saving
90	0.25	\$3,470
70	0.40	2,920
50	0.25	2,310
30	0.10	1,560

Based on a before-tax analysis (MARR = 18% per year, analysis period = 6 years, and market value = 0) and E(PW) as the decision criterion, is the new compressor design economically preferable to the current unit?

Solution

The single-stage decision tree diagram for the design alternatives is shown in Figure 12-10. The PWs associated with each of the efficiency design goal levels

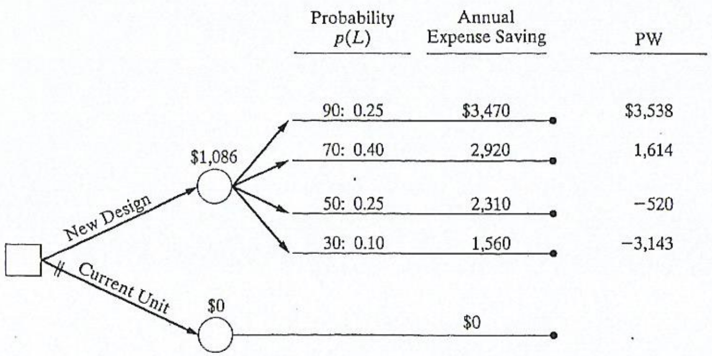


Figure 12-10 Single-Stage Decision Tree (Example 12-10)

being met are as follows:

$$\text{PW}(18\%)_{90} = -\$8{,}600 + \$3{,}470(P/A, 18\%, 6) = \$3{,}538$$

$$\text{PW}(18\%)_{70} = -\$8{,}600 + \$2{,}920(P/A, 18\%, 6) = \$1{,}614$$

$$\text{PW}(18\%)_{50} = -\$8{,}600 + \$2{,}310(P/A, 18\%, 6) = -\$520$$

$$\text{PW}(18\%)_{30} = -\$8{,}600 + \$1{,}560(P/A, 18\%, 6) = -\$3{,}143$$

Based on these values, the E(PW) of each installed unit of the new compressor is as follows:

$$E(\text{PW}) = 0.25(\$3{,}538) + 0.40(\$1{,}614) + 0.25(-\$520) + 0.10(-\$3{,}143) = \$1{,}086.$$

The E(PW) of the current unit is zero since the cash-flow estimates for the new design are incremental amounts relative to the present design. Therefore, the analysis indicates that the new design is economically preferable to the present design. (The parallel lines across the current unit path on the diagram indicate that it was not selected.)

12.9 Real Options Analysis

Companies make capital investments to exploit opportunities for shareholder (owner) wealth creation. Often these opportunities are *real options,* which are opportunities available to a firm when it invests in real assets such as plant, equipment, and land. Real options allow decision makers to invest capital now or to postpone all or part of the investment until later. Decision trees are a convenient means for portraying the resolution of risk/uncertainty in the analysis of real options.*

The real options approach to capital investments is based on an interesting analogy about financial options. A company with an opportunity to invest capital actually owns something much like a financial call option—the company has the right but not the obligation to invest in (purchase) an asset at a future time of its choosing. When a firm makes an irreversible capital investment that could be postponed, it exercises its call option, which has value by virtue of the flexibility it gives the firm.

* The reader interested in details regarding real options analysis is referred to these two articles: Luehrman, T. A., "Investment Opportunities as Real Options: Getting Started with the Numbers," *Harvard Business Review,* 76, no. 4 (July–August 1998a): 51–67; Luehrman, T. A., "Strategy as a Portfolio of Real Options," *Harvard Business Review,* 76, no. 5 (September–October 1998b): 89–99.

An example of a postponable investment is coal-fired generating capacity of an electric utility. Anticipated capacity needed for the next 10 years can be added in one large expansion project, or the capacity addition can be acquired in staggered stages, which permits the utility to better respond to future demand patterns and possibly different types of generating capacity, such as natural gas or nuclear power. If the utility company decides to go ahead with a single, large, irreversible expansion project, it eliminates the option of waiting for new information that might represent a more valuable phased approach to meeting customers' demands for electricity. The lost option's value is an *opportunity cost* that must be included in the overall evaluation of the investment. This is the essence of the real options approach to capital investment—to fairly value the option of waiting to invest in all or a part of the project and to include this value in the overall project profitability. Clearly, viewing capital investments as real options forces a greater emphasis to be placed on the value of information in risky situations facing a firm.

In the previous section, we have shown how decision trees are useful in determining the value of information of anticipated future outcomes. In this regard, decision trees enable the analyst to model real options that are hidden in classical present worth analysis. These options, not treated formally in classic single outcome analyses, are often understood by management and factored informally into their assessment of the analysis (and the assessment of the analyst!). This presence of hidden options is most easily understood with an example.

EXAMPLE 12-11

Hytech Industries

Hytech is considering opening an entirely new electronic interface using radio frequency identification with a new coding system that deviates from some standards but offers advantages to bulk chemical manufacturers. The cost to develop the manufacturing capacity and build the market is estimated at \$700 million, and the resulting net cash inflows after income taxes are estimated at \$100 million per year (albeit with much potential variation depending upon acceptance of the product). The after-tax MARR is 15% per year, and the new product will have a life of 20 years.

Solution as Classical Single Outcome Analysis

The PW of this project is

$$\text{PW}(15\%) = -\$700 \text{ million} + \$100 \text{ million}(P/A, 15\%, 20) = -\$74.1 \text{ million},$$

and the recommendation would be to avoid the investment.

Suppose, however, that the \$100 million cash inflow per year is merely an average. If the device catches on, potential sales will exceed the annual capacity for the new facilities, and annual inflows will be limited to \$200 million. In fact, if this happens, a second plant can be acquired one year later for an additional investment of \$500 million, and it would be able to generate an additional \$150 million cash inflow annually. On the other hand, if the new product fails to catch on, sales could effectively be zero, but the plant could be salvaged for \$150 million.

Solution as a Decision Tree with Embedded Options

The decision tree appears as shown in Figure 12-11. If the end-of-path convention is used, then the tip values are as shown in Table 12-16. Note that path 3, abandon when first year inflows exceed \$200 million, is not reasonable and is not evaluated. Similarly, path 4 and 7 would involve expansion when the market has not developed and are not evaluated.

As shown in the table, when first year net inflows are high (\$200 million +), the better of paths 1 and 2 is to expand and realize a PW of \$926 million. If response is tepid (\$100 million inflow), then path 5 is better than path 6. The PW of path 5 is negative, but staying in business recoups some of the investment and is better than abandoning the project. Finally, if the market is negligible, path 9 is better than path 8, because the ability to salvage some of the investment mitigates the loss.

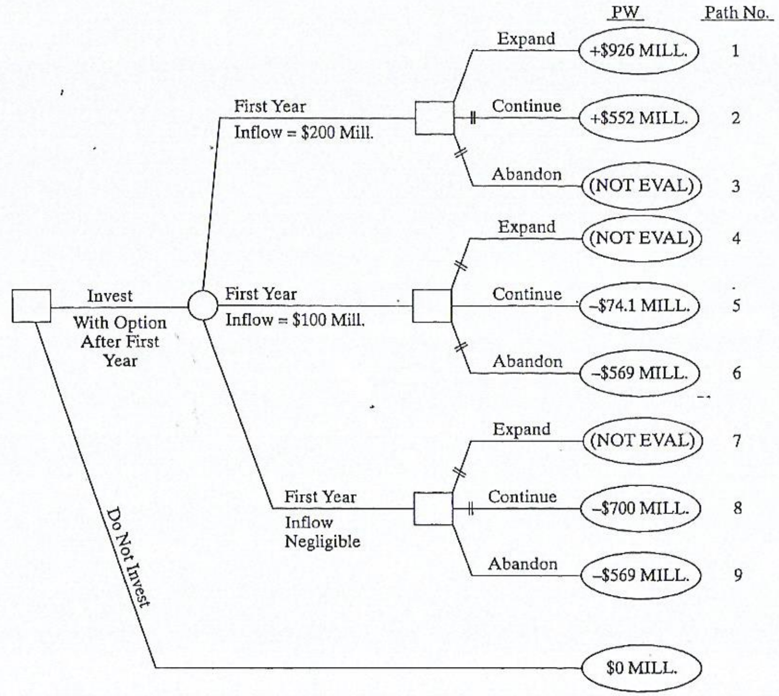


Figure 12-11 Hytech Industries Example

Decision Tree Endpoint Evaluation

Path	First Year Inflows	Option	Evaluation	Net PW
1	$200 million +	Expand	$-\$700 + \$200(P/A, 15\%, 20)$ $-\$500(P/F, 15\%, 1)$ $+\$150(P/A, 15\%, 19)(P/F, 15\%, 1)$	+$926 million
2	$200 million +	Continue	$-\$700 + \$200(P/A, 15\%, 20)$	+$552 million
3	$200 million +	Abandon	Not evaluated	
4	$100 million	Expand	Not evaluated	
5	$100 million	Continue	$-\$700 + \$100(P/A, 15\%, 20)$	−$74.1 million
6	$100 million	Abandon	$-\$700 + \$150(P/F, 15\%, 1)$	−$569 million
7	Negligible	Expand	Not evaluated	
8	Negligible	Continue	−$700	−$700 million
9	Negligible	Abandon	$-\$700 + \$150(P/F, 15\%, 1)$	−$569 million

If each outcome of the first year cash flows were equally likely and management is risk neutral, then the expected monetary value is $94.3 (= [+$926 − $74.1 − $569]/3) million, an improvement of $168.4 million over the $74.1 million loss projected in the single outcome analysis. This improvement can be viewed as the value of the combined options to expand, if the market develops, or to salvage some investment if the project is a flop.

12.10 Summary

Engineering economy involves decision making among competing uses of scarce capital resources. The consequences of resultant decisions usually extend far into the future. In this chapter, we have presented various statistical and probability concepts that address the fact that the consequences (cash flows, project lives, etc.) of engineering alternatives can never be known with certainty, including Monte Carlo computer simulation techniques and decision tree analysis. Cash inflow and cash outflow factors, as well as project life, were modeled as discrete and continuous random variables. The resulting impact of uncertainty on the economic measures of merit for an alternative was analyzed. Included in the discussion were several considerations and limitations relative to the use of these methods in application.

Regrettably, there is no quick and easy answer to the question, How should risk best be considered in an engineering economic evaluation? Generally, simple procedures (e.g., breakeven analysis and sensitivity analysis, discussed in Chapter 11) allow some discrimination among alternatives to be made on the basis of the uncertainties present, and they are relatively inexpensive to apply. Additional discrimination among alternatives is possible with more complex procedures that utilize probabilistic concepts. These procedures, however, are more difficult to apply and require additional time and expense.

Problems

The number in parentheses that follows each problem refers to the section from which the problem is taken.

12-1. A large mudslide caused by heavy rains will cost Sabino County $1,000,000 per occurrence in lost property tax revenues. In any given year, there is one chance in 100 that a major mudslide will occur.

A civil engineer has proposed constructing a culvert on a mountain where mudslides are likely. This culvert will reduce the likelihood of a mudslide to near zero. The investment cost would be $50,000, and annual maintenance expenses would be $2,000 in the first year, increasing by 5% per year thereafter. If the life of the culvert is expected to be 20 years and the cost of capital to Sabino County is 7% per year, should the culvert be built? **(12.4)**

12-2. A bridge is to be constructed now as part of a new road. An analysis has shown that traffic density on the new road will justify a two-lane bridge at the present time. Because of uncertainty regarding future use of the road, the time at which an extra two lanes will be required is currently being studied. The estimated probabilities of having to widen the bridge to four lanes at various times in the future are as follows:

Widen Bridge In	Probability
3 years	0.1
4 years	0.2
5 years	0.3
6 years	0.4

The present estimated cost of the two-lane bridge is $2,000,000. If constructed now, the four-lane bridge will cost $3,500,000. The future cost of widening a two-lane bridge will be an extra $2,000,000 plus $250,000 for every year that widening is delayed. If money can earn 12% per year, what would you recommend? **(12.4)**

12-3. In Problem 12-2, perform an analysis to determine how sensitive the choice of a four-lane bridge built now versus a four-lane bridge that is constructed in two stages is to the interest rate. Will an interest rate of 15% per year reverse the initial decision? At what interest rate would constructing the two-lane bridge now be preferred? **(12.4)**

12-4. A machine requires a capital investment of $200,000 and operating expenses will be 20% of the revenue from the sale of the product produced using the machine. It is found that the machine is highly productive and produces 1,000,000 products per year. The probability of purchasing the machine is about 10%. **(12.4)**

a. If the product is sold for $20 per unit, what is the E(PW) of profit to the owner/operator of this machine? The life of the machine is 10 years and MARR is 12% per year.

b. Repeat Part (a) when the life of the machine is eight years.

c. Perform one-at-a-time sensitivity analyses for ±20 per cent changes in yearly production and selling price of the product. Use a 10-year life for the machine.

12-5. The cost of the fuel (C) to run a vehicle is a function of the distance travelled (S) and fuel consumption rate (R); that is, $C = R.S$. It is desired to know the probability distribution of the random variable C when R and S have the following assumed probability mass functions (R and S are independent):

R (Fuel Consumption Rate, liter/mile)		Distance Moved	
Value (liter)	Probability	Value (mile)	Probability
5	0.1	50	0.25
10	0.25	60	0.35
20	0.28	70	0.20
30	0.30		

What are the mean, variance, and standard deviation of the probability distribution for annual cost of fuel, C? **(12.4)**

12-6. A very important levee spans a distance of 10 miles on the outskirts of a large metropolitan area. Hurricanes have hit this area twice in the past 20 years, so there is concern over the structural integrity of this particular levee. City engineers have proposed reinforcing and increasing the height of the levee by various designs to withstand the storm surge of numerous strength categories of hurricanes.

Study results regarding the probability that high flood water from a hurricane in any one year will exceed the increased height of the levee, and the cost of construction of the levee for each storm category, are summarized next.

Hurricane Category	Probability of a Storm Surge Exceeding the Levee Height	Capital Investment to Increase the Levee Height
5	0.005	$70,000,000
4	0.010	50,000,000
3	0.030	35,000,000
2	0.050	20,000,000
1	0.100	10,000,000

A panel of experts suggests that the average property damage will amount to $100,000,000 if a storm surge causes the levee to overflow. The capital investment to rebuild the levee for each hurricane category will be financed with 30-year municipal bonds earning 6% per year. These bonds will be retired as annuity payments each year. What is the most economical way to rebuild the levee to protect the city from flooding during a hurricane? What other factors might affect the decision in this situation? What if the average damage from a flood is $200,000,000? **(12.4)**

12-7. A diesel generator is needed to provide auxiliary power in the event that the primary source of power is interrupted. Various generator designs are available, and more expensive generators tend to have higher reliabilities should they be called on to produce power. Estimates of reliabilities, capital investment costs, operating and maintenance expenses, market value, and damages resulting from a complete power failure (i.e., the standby generator fails to operate) are given in Table P12-7 for three alternatives. If the life of each generator is 10 years and MARR = 10% per year, which generator should be chosen if you assume one main power failure per year? Does your choice change if you assume two main power failures per year? (Operating and maintenance expenses remain the same.) **(12.4)**

12-8. The owner of a ski resort is considering installing a new ski lift that will cost $900,000. Expenses for operating and maintaining the lift are estimated to be $1,500 per day when operating. The U.S. Weather Service estimates that there is a 60% probability of 80 days of skiing weather per year, a 30% probability of 100 days per year, and a 10% probability of 120 days per year. The operators of the resort estimate that during the first 80 days of adequate snow in a season, an average of 500 people will use the lift each day, at a fee of $10 each. If 20 additional days are available, the lift will be used by only 400 people per day during the extra period; and if 20 more days of skiing are available, only 300 people per day will use the lift during those days. The owners wish to recover any invested capital within five years and want at least a 25% per year rate of return before taxes. Based on a before-tax analysis, should the lift be installed? **(12.4)**

12-9. Refer to Problem 12-8. Assume the following changes: the study period is eight years; the ski lift will be depreciated by using the MACRS Alternative Depreciation System (ADS); the ADS recovery period is seven years; MARR = 15% per year (after-tax); and the effective income tax rate (t) is 40%. Based on this information, what is the E(PW) and SD(PW) of the ATCF? Interpret the analysis results and make a recommendation on installing the ski lift. **(12.4)**

12-10. An energy conservation project is being evaluated. Four levels of performance are considered feasible. The estimated probabilities of each performance level and the estimated before-tax cost savings in the first year are shown in the following table:

Performance Level (L)	$p(L)$	Cost Savings (1st yr; before taxes)
1	0.15	$22,500
2	0.25	35,000
3	0.35	44,200
4	0.25	59,800

TABLE P12-7 Three Generator Designs for Problem 12-7

Alternative	Capital Investment	O&M Expenses/Year	Reliability	Cost of Power Failure	Market Value
R	$200,000	$5,000	0.96	$400,000	$40,000
S	170,000	7,000	0.95	400,000	25,000
T	214,000	4,000	0.98	400,000	38,000

Assume the following:

- Initial capital investment: $100,000 [80% is depreciable property and the rest (20%) are costs that can be immediately expensed for tax purposes].
- The ADS under MACRS is being used. The ADS recovery period is four years.
- The before-tax cost savings are estimated to increase 6% per year after the first year.
- MARR_{AT} = 12% per year; the analysis period is five years; $MV_5 = 0$.
- The effective income tax rate is 40%.

Based on E(PW) and after-tax analysis, should the project be implemented? **(12.4)**

12-11. A new machine is purchased to be part of a manufacturing process in order to increase efficiency and the rate of production. The machine costs $500,000 and the estimated savings from using the machine are $200,000 per year. The useful life of the equipment in this application is uncertain. The estimated probabilities of different useful lives occurring are shown in the following table. Assume that MARR = 12% per year before taxes and the market value at the end of its useful life is equal to zero. Based on a before-tax analysis

a. What are the E(PW), V(PW), and SD(PW) associated with the purchase of the equipment? **(12.4)**

b. What is the probability that the PW is less than zero? Make a recommendation and give your supporting logic based on the results of your analysis.

Useful Life, years (N)	$p(N)$
1	0.1
2	0.1
3	0.3
4	0.2
5	0.2
6	0.1

12-12. The tree diagram in Figure P12-12 describes the uncertain cash flows for an engineering project. The analysis period is two years, and MARR = 15% per year. Based on this information,

a. What are the E(PW), V(PW), and SD(PW) of the project?

b. What is the probability that PW ≥ 0? **(12.3)**

12-13. A potential project has an initial capital investment of $100,000. Net annual revenues minus expenses are estimated to be $40,000 (A$) in the first year and to increase at the rate of 6.48% per year. The useful life of the primary equipment, however, is uncertain, as shown in the following table:

Useful Life, Years (N)	$p(N)$
1	0.03
2	0.10
3	0.30
4	0.30
5	0.17
6	0.10

Assume that i_m = MARR = 15% per year and f = 4% per year. Based on this information,

a. What are the E(PW) and SD(PW) for this project?

b. What is the Pr[PW ≥ 0]?

c. What is the E(AW) in R$?

Do you consider the project economically acceptable, questionable, or not acceptable, and why? **(Chapter 8, 12.4)**

12-14. A hospital administrator is faced with the problem of having a limited amount of funds available for capital projects. He has narrowed his choice down to two pieces of x-ray equipment, since the radiology department is his greatest producer of revenue. The first piece of equipment (Project *A*) is a fairly standard piece of equipment that has gained wide acceptance and should provide a steady flow of income. The other piece of equipment (Project *B*), although more risky, may provide a higher return. After deliberation with his radiologist and director of finance, the administrator has developed the following table:

Cash Flow per Year (thousands)			
Probability	Project *A*	Probability	Project *B*
0.6	$2,000	0.2	$4,000
0.3	1,800	0.5	1,200
0.1	1,000	0.3	900

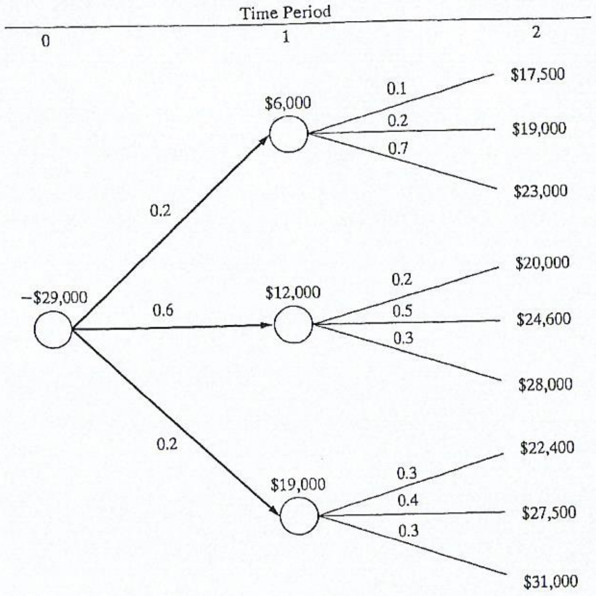


Figure P12-12 Probability Tree Diagram for Problem 12-12

Discovering that the budget director of the hospital is taking courses in engineering, the hospital administrator has asked him to analyze the two projects and make his recommendation. Prepare an analysis that will aid the budget director in making his recommendation. In this problem, do risk and reward travel in the same direction? **(12.4)**

12-15. In a construction company, project completion time is considered to be normally distributed with mean completion time being 16 weeks. There is an 80% probability that a project will take between 14 to 18 weeks to reach completion. What is the variance of project completion time? **(12.5)**

12-16. In the production of shafts, it is assumed that the diameters of the shafts produced are normally distributed, with mean = 50mm and variance = 8mm^2. What is the probability that the diameter of a shaft will be at least 48 mm? **(12.5)**

12-17. The use of three estimates (defined here as H = high, L = low, and M = most likely) for random variables is a practical technique for modeling uncertainty in some engineering economy studies. Assume that the mean and variance of the random variable, X_k, in this situation can be estimated by $E(X_k) = (1/6)(H + 4M + L)$ and $V(X_k) = [(H - L)/6]^2$. The estimated net cash-flow data for one alternative associated with a project are shown in Table P12-17.

The random variables, X_k, are assumed to be statistically independent, and the applicable MARR = 15% per year. Based on this information,

a. What are the mean and variance of the PW?

b. What is the probability that PW $\geq$ 0 (state any assumptions that you make)?

c. Is this the same as the probability that the IRR is acceptable? Explain. **(12.5)**

Estimates for Problem 12-17

End of Year, k	Net Cash Flow	Three-Point Estimates for X_k: L	M	H
0	$F_0 = X_0$	−$38,000	−$41,000	−$45,000
1	$F_1 = 2X_1$	−1,900	−2,200	−2,550
2	$F_2 = X_2$	9,800	10,600	11,400
3	$F_3 = 4X_3$	5,600	6,100	6,400
4	$F_4 = 5X_4$	4,600	4,800	5,100
5	$F_5 = X_5$	16,500	17,300	18,300

12-18. Two mutually exclusive investment alternatives are being considered. Alternative A requires an initial investment of $20,000 in a machine. Annual operating and maintenance costs are anticipated to be normally distributed, with a mean of $8,000 and a standard deviation of $600. The salvage value at the end of its life (10 years) is anticipated to be normally distributed, with a mean of $2,000 and a standard deviation of $900.

Alternative B requires end-of-year annual expenditure over the ten-year planning horizon, with the annual expenditure being normally distributed with a mean of $10,500 and a standard deviation of $1,200. Using a MARR of 12% per year, what is the probability that alternative A is the most economic alternative (i.e., the least costly)? **(12.5)**

12-19. Two investment options are to be compared. The data below have been estimated by a committee of experts, and all the cash flows are assumed to be independent. Life is *not* a variable. With MARR = 15% per year, determine the mean and standard deviation of the *incremental* PW [i.e., Δ(B-A)]. **(12.5)**

End of Year	Alternative A: Expected Cash Flow	Alternative A: Std. Deviation of Cash Flow	Alternative B: Expected Cash Flow	Alternative B: Std. Deviation of Cash Flow
0	−$10,000	$0	−$15,000	$700
1	$2,500	$400	$5,000	$500
2	$5,000	$500	$5,000	$500
3	$3,000	$550	$5,000	$500
4	$6,500	$600	$5,000	$500

12-20. The market value of an asset depends upon its useful life as follows:

Useful Life, year	Market Value
5	$20,000
6	$16,000
7	$10,000

a. The asset requires a capital investment of $150,000 and MARR is 12% per year. Use Monte Carlo simulation and generate four trial outcomes to find its expected equivalent AW of each useful life is equally likely to occur. **(12.6)**

b. Set up an equation to determine the variation of the asset's AW. **(12.5)**

12-21. A company that manufactures automobile parts is weighing the possibility of investing in an FMC (Flexible Manufacturing Cell). This is a separate investment and not a replacement of the existing facility. Management desires a good estimate of the distribution characteristics for the AW. There are three random variables; i.e., they have uncertainties in their values.

An economic analyst is hired to estimate the desired parameter, AW. He concludes that the scenario presented to him is suitable for Monte Carlo simulation (method of statistical trials) because there are uncertainties in three variables and the direct analytical approach is virtually impossible.

The company has done a preliminary economic study of the situation and provided the analyst with the following estimates:

Investment	Normally distributed with mean of $200,000 and standard deviation of $10,000
Life	Uniformly distributed with minimum of 5 years and maximum of 15 years
Market value	$20,000 (single outcome)
Annual net cash flow	$20,000 probability 0.3 $16,000 probability 0.5 $22,000 probability 0.2
Interest rate	10%

All the elements subject to variation vary independently. Use Monte Carlo simulation to generate 10 AW outcomes for the proposed FMC. What is the standard deviation of the outcomes? **(12.6, 12.7)**

12-22. Simulation results are available for two mutually exclusive alternatives. A large number of trials have been run with a computer, with the results shown in Figure P12-22. Discuss the issues that may arise when attempting to decide between these two alternatives. **(12.6, 12.7)**

12-23. If the interest rate is 8% per year, what decision would you make based on the decision tree diagram in Figure P12-23? **(12.8)**

12-24. *Extended Learning Exercise* The additional investment in a new computer system is a certain $300,000. It is likely to save an average of $100,000 per year compared to the old, outdated system. Because of uncertainty, this estimate is expected to be normally distributed, with a standard deviation of $7,000. The market value of the system at any time is its scrap value, which is $20,000 with a standard deviation of $3,000. MARR on such investments is 15% per year.

What is the smallest value of N (the life of the system) that can exist such that the probability of getting a 15% internal rate of return or greater is 0.90? Ignore the effects of income taxes. Also note that market value is independent of N. (*Hint*: Start with $N = 4$ years.) **(12.4)**

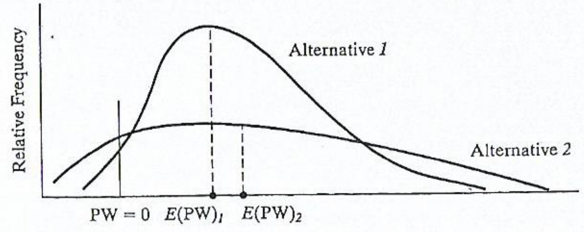


Figure P12-22 Simulation Results for Problem 12-22

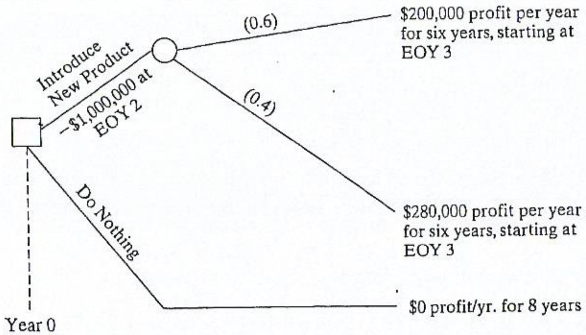


Figure P12-23 Decision Tree Diagram for Problem 12-23

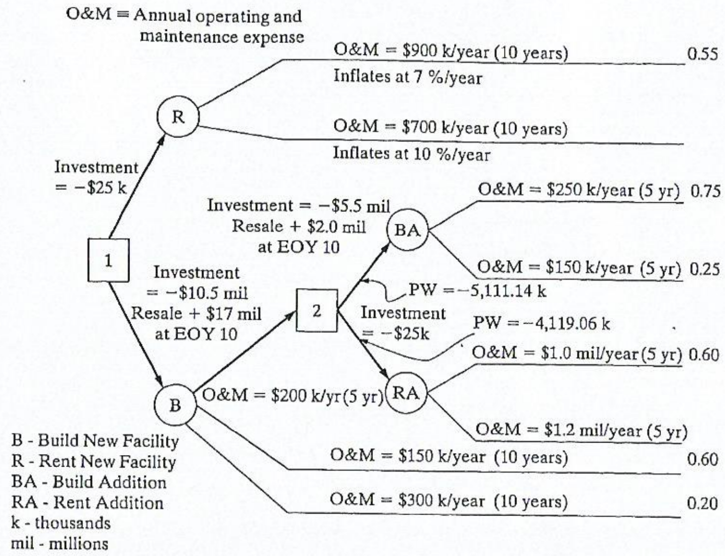


Figure P12-25 Decision Tree Diagram for Problem 12-25

12-25. *Extended Learning Exercise* A firm must decide between constructing a new facility or renting a comparable office space. There are two random outcomes for acquiring space, as shown in Figure P12-25. Each would accommodate the expected growth of this company over the next 10 years. The cost of rental space is expected to escalate over the 10 years for each rental outcome.

The option of constructing a new facility is also defined in Figure P12-25. An initial facility could be constructed with the costs shown. In five years, additional space will be required. At that time, there will be an option to build an office addition or rent space for the additional space requirements.

The probabilities for each alternative are shown. MARR for the situation is 10% per year. A PW analysis is to be conducted on the alternatives. Which course of action should be recommended? Note: At [2], the PW(10%) of the upper branch is −$5,111.14 k and the PW(10%) for the lower branch is −$4,119.06 k.

Spreadsheet Exercises

12-26. Refer to Example 12-4. After additional research, the range of probable useful lives has been narrowed down to the following three possibilities:

Useful Life, Years (N)	$p(N)$
14	0.3
15	0.4
16	0.3

How does this affect $E(PW)$ and $SD(PW)$? What are the benefits associated with the cost of obtaining more accurate estimates? **(12.4)**

12-27. Refer to Problem 12-21. Use a spreadsheet to extend the number of trials to 500. Compute $E(AW)$ and plot the cumulative average AW. Are 500 trials enough to reach steady state? **(12.7)**